Discrete dynamical systems associated with root systems of indefinite type

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Abstract

A geometric approach to the equation found by Hietarinta and Viallet, which satisfies the singularity confinement criterion but which exhibits chaotic behavior, is presented. It is shown that this equation can be lifted to an automorphism of a certain rational surface and can therefore be considered to be the action of an extended Weyl group of indefinite type. A method to construct the mappings associated with some indefinite root systems is presented. A method to calculate their algebraic entropy by using the theory of intersection numbers is presented. It is also shown that the degree of the n-th iterate of every discrete Painlevé equation in Sakai's list is at most $O(n^2)$ and therefore its algebraic entropy is zero.

1 Introduction

The singularity confinement method has been proposed by Grammaticos, Ramani and Papageorgiou [1] as a criterion for the integrability of (finite or infinite dimensional) discrete dynamical systems. The singularity confinement method demands that when singularities appear due to particular initial values such singularities should disappear after a finite number of iteration steps, in which case the information on the initial values ought to be recovered (hence the dynamical system has to be invertible).

However "counter examples" were found by Hietarinta and Viallet [2]. These mappings satisfy the singularity confinement criterion but the orbits of their solutions exhibit chaotic behavior. The authors of [2] introduced the notion of algebraic entropy in order to test the degree of complexity of successive iterations. The algebraic entropy is defined as $s := \lim_{n\to\infty} \log(d_n)/n$ where d_n is the degree of the n-th iterate. This notion is linked to Arnold's complexity since the degree of a mapping gives the intersection number of the image of a line and a hyperplane. While the degree grows exponentially for a generic mapping, it was shown that it only grows polynomially for a large class of integrable mappings [2, 3, 4, 5].

The discrete Painlevé equations have been extensively studied [6, 7]. Recently it was shown by Sakai [8] that all of (from the point of view of symmetries) these are obtained by studying rational surfaces in connection with the extended affine Weyl groups.

Surfaces obtained by successive blow ups [9] of $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ have been studied by several authors by means of connections between the Weyl groups and the groups of Cremona isometries on the Picard group of the surfaces [10, 11, 12]. Here, the Picard group of a rational surface X is the group of isomorphism classes of invertible sheaves on X and it is isomorphic to the group of linear equivalent classes of divisors on X. A Cremona isometry is an isomorphism of the Picard group such that a) it preserves the intersection number of any pair of divisors, b) it preserves the canonical divisor K_X and c) it leaves the set of effective classes of divisors invariant. In the case where 9 points (in the case of $\mathbb{P}^2$, 8 points in the case of $\mathbb{P}^1 \times \mathbb{P}^1$) are blown up, if the points are in general position the group of Cremona isometries becomes isomorphic with an extension of the Weyl group of type $E_8^{(1)}$. In case the 9 points are not in general position, the classification of connections between the group of Cremona isometries and the extended affine Weyl groups was first studied by Looijenga [13] and more generally by Sakai. Birational (bimeromorphic) mappings on $\mathbb{P}^2$ (or $\mathbb{P}^1 \times \mathbb{P}^1$) are obtained by interchanging the procedure of blow downs. Discrete Painlevé equations are recovered as the birational mappings corresponding to the translations of affine Weyl groups.

Our aim in this paper is to characterize some birational mappings which satisfy the singularity confinement criterion but exhibit chaotic behavior from the point of view of the theory of rational surfaces. Considering one such mapping and the space of its initial values, we obtain a rational surface associated with some root system of hyperbolic type. Conversely, we recover the mapping from the surface and consequently obtain the extension of the mapping to its non-autonomous version. It is important to remark that this method also allows the construction of other mappings starting from suitable extended Weyl groups. We also show some other examples of such constructions. We also present a method to calculate the degree of the n-th iterate for a mapping. We show that for all discrete Painlevé equations, the degrees grow at most as $O(n^2)$.

In Section 2, we start from one of the mappings found by Hietarinta and Viallet (we call it the HV eq. in this paper) and construct the space such that the mappings is lifted to an automorphism, i.e. bi-holomorphic mapping, of the surface. The mapping φ' is called a mapping lifted from the mapping φ if φ' coincides with φ on any point where φ is defined. For this purpose we compactify the original space of initial values, $\mathbb{C}^2$, to $\mathbb{P}^1 \times \mathbb{P}^1$ and blow up 14 times.

In Section 3, we study the symmetry of the space of initial values. We show that the group of all the Cremona isometries of the Picard group of the surface is isomorphic to an extended Weyl group of hyperbolic type. As a corollary, we prove that there does not exist any Cremona isometry whose action on the Picard group commutes with the action of the HV eq. except the action itself.

In Section 4, we show a method to recover the HV eq. from the surface as an element of the extended Weyl group. Each element of the extended Weyl group which acts on the Picard group as a Cremona isometry, is realized as a Cremona transformation (i.e. a birational mapping) on $\mathbb{P}^1 \times \mathbb{P}^1$ by interchanging the blow down structure. Here, a blow down structure is the sequence designating the procedure of blow downs. As a result of this we obtain the non-autonomous version of the equation.

In Section 5, we show a method to calculate the degree of the n-th iterate of the mapping which is lifted to an isomorphism of a suitable rational surface. Considering the intersection numbers of divisors it is shown that the degree is given by the n-th power of a matrix given by the action of the mapping on the Picard group. Applying this method for discrete Painlevé equations, we show the degrees of the n-th iterate are at most $O(n^2)$.

In Section 6, we discuss the construction of other mappings from certain Weyl groups and show some examples.

2 Construction of the space of initial values by blow ups

We consider the dynamical system written by the birational (bi-meromorphic) mapping

$$\begin{array}{rccl} \varphi: & \mathbb{C}^2 & \rightarrow & \mathbb{C}^2 \\ & \begin{pmatrix} x_n \\ y_n \end{pmatrix} & \mapsto & \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} y_n \\ -x_n + y_n + a/y_n^2 \end{pmatrix} \end{array} \tag{1}$$

where $a \in \mathbb{C}$ is a nonzero constant. This equation was found by Hietarinta and Viallet [2] and we call it the HV eq.. To test the singularity confinement, let us assume $x_0 \neq 0$ and $y_0 = \epsilon$ where $|\epsilon| \ll 1$. With these initial values we obtain the sequence:

$$\begin{array}{lcl} x_0 & = & x_0 \\ x_1 = y_0 & = & \epsilon \\ x_2 = y_1 & = & a\epsilon^{-2} - x_0 + \epsilon \\ x_3 = y_2 & = & a\epsilon^{-2} - x_0 + a^{-1}\epsilon^4 + O(\epsilon^6) \\ x_4 = y_3 & = & -\epsilon + 2a^{-1}\epsilon^4 + 4x_0a^{-2}\epsilon^6 + O(\epsilon^7) \\ x_5 = y_4 & = & x_0 + 3\epsilon + O(\epsilon^2) \\ x_6 = y_5 & = & (ax_0^{-2} + x_0) + O(\epsilon) \\ & \vdots & . \end{array}$$

In this sequence singularities appear at $n = 1$ as $\epsilon \to 0$ and disappear at $n = 4$ and the information on the initial values is hidden in the coefficients of higher degree ϵ. However, taking suitable rational functions of x_n and y_n we can find the information of the initial values as finite values. The fact that the leading orders of $(x_1^2y_1 - a)y_1$, $(x_2^3(y_2/x_2 - 1)^2 - a)x_2$ and $(x_3y_3^2 - a)x_3$ become $-ax_0, -ax_0$ and $-ax_0$ actually suggests that the HV eq. can be lifted to an automorphism of a suitable rational surface (although these rational functions are of course not uniquely determined).

Let us consider the HV eq. φ to be a mapping from the complex projective space $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ $(= \mathbb{P}^1 \times \mathbb{P}^1)$ to itself. We use the terminology *space of initial values* as follows (analogous to the space of initial values of Painlevé equations introduced by Okamoto[14]).

Definition. . A sequence of algebraic varieties X_i is (or X_i themselves are) called the space of initial values for the sequence of rational mappings φ_i, if each φ_i is lifted to an isomorphism from X_i to X_{i+1}, for all i.

Our aim in this section is to construct the surface X by blow ups $\mathbb{P}^1 \times \mathbb{P}^1$ such that φ is lifted to an automorphism of X, where the mapping φ' is called a mapping lifted from the mapping φ if φ' coinsides with φ on any point where φ is defined.

2.1 Regular mapping from Y_1 to $\mathbb{P}^1 \times \mathbb{P}^1$

Let the coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$ be $(x, y), (x, 1/y), (1/x, y)$ and $(1/x, 1/y)$ and let $x = \infty$ denote $1/x = 0$. We denote the HV eq. as

$$\varphi \quad : \quad (x, y) \mapsto (\overline{x}, \overline{y}) = (y, -x + y + a/y^2) \tag{2}$$

where $(\overline{x}, \overline{y})$ means the image of (x, y) by the mapping. This mapping has two indeterminate points: $(x, y) = (\infty, 0)$, (∞, ∞). By blowing up at these points we can ease the indeterminacy.

We denote blowing up at $(x, y) = (x_0, y_0) \in \mathbb{C}^2$:

$$\begin{aligned} & \{(x, y) : x, y \in \mathbb{C}\} \\ \xleftarrow{\mu_{(x_0,y_0)}} \quad & \{(x - x_0,\ y - y_0;\ \zeta_1 : \zeta_2) \mid x, y, \zeta_1, \zeta_2 \in \mathbb{C} \wedge (y - y_0)\zeta_1 = (x - x_0)\zeta_2\} \\ = \quad & \{(x - x_0, \zeta_2/\zeta_1 \mid x, \zeta_1, \zeta_2 \in \mathbb{C} \wedge \zeta_1 \neq 0\} \cup \{(\zeta_1/\zeta_2,\ y - y_0 \mid y, \zeta_1, \zeta_2 \in \mathbb{C} \wedge \zeta_2 \neq 0\} \end{aligned}$$

by

$$(x, y) \leftarrow (x - x_0, (y - y_0)/(x - x_0)) \cup ((x - x_0)/(y - y_0), y - y_0). \tag{3}$$

In this way, blowing up at $(x, y) = (x_0, y_0)$ gives meaning to $(x-x_0)/(y-y_0)$ at this point.

First we blow up at $(x, y) = (\infty, 0)$, $(1/x, y) \leftarrow (1/x, xy) \cup (1/xy, y)$ and denote the obtained surface by Y_0. Then φ is lifted to a rational mapping from Y_0 to $\mathbb{P}^1 \times \mathbb{P}^1$. For example, in the new coordinates φ is expressed as

$$\begin{aligned} (u_1, v_1) := (1/x, xy) \quad &\mapsto \quad (\overline{x}, \overline{y}) = (u_1 v_1, (-u_1 v_1^2 + u_1^3 v_1^3 + a)/(u_1^2 v_1^2)) \\ (u_2, v_2) := (1/xy, y) \quad &\mapsto \quad (\overline{x}, \overline{y}) = (v_2, (-v_2 + u_2 v_2^3 + a u_2)/(u_2 v_2^2)) \end{aligned}$$

This maps the exceptional curve at $(x, y) = (\infty, 0)$, i.e.$u_1 = 0$ and $v_2 = 0$, almost to $(\overline{x}, \overline{y}) = (\infty, 0)$ but has an indeterminate point on the exceptional curve: $(u_2, v_2) = (0, 0)$. Hence we have to blow up again at this point. In general it is known that, if there is a rational mapping $X \to X'$ where X and X' are smooth projective algebraic varieties, the procedure of blowing up can be completed in a finite number of steps, after which one obtains a smooth projective algebraic variety Y such that the rational mapping is lifted to a regular mapping from Y to X' (theorem of the elimination of indeterminacy [9]).

Here we obtain the surface Y_1 defined by the following sequence of blow ups. (For simplicity we take only one coordinate of (3).)

$$\begin{aligned} (x, y) \quad & \xleftarrow[\mu_1]{(\infty,0)} \quad \left(\frac{1}{xy}, y\right) \xleftarrow[\mu_2]{(0,0)} \left(\frac{1}{xy}, xy^2\right) \\ & \xleftarrow[\mu_3]{(0,a)} \quad \left(\frac{1}{xy}, xy(xy^2 - a)\right) \xleftarrow[\mu_4]{(0,0)} \left(\frac{1}{xy}, x^2y^2(xy^2 - a)\right) \\ (x, y) \quad & \xleftarrow[\mu_9]{(\infty,\infty)} \quad \left(\frac{1}{x}, \frac{x}{y}\right) \xleftarrow[\mu_{10}]{(0,1)} \left(\frac{1}{x}, x(\frac{x}{y} - 1)\right) \end{aligned}$$

where μ_i denotes the i-th blow up. Of course the above sequence is not unique since there is freedom in choosing the coordinates.

2.2 Automorphism of X

We have obtained a mapping from Y_1 to $\mathbb{P}^1 \times \mathbb{P}^1$ which is lifted from φ. But our aim is to construct a rational surface X such that φ is lifted to an automorphism of X.

First we construct the rational surface Y_2 such that φ is lifted to a regular mapping from Y_2 to Y_1. For this purpose it is sufficient to eliminate the indeterminacy of mapping from Y_1 to Y_1. Consequently we obtain Y_2 defined by the following sequence of blow ups.

$$\begin{aligned}
\left(\frac{1}{x}, x(\frac{x}{y}-1)\right) \quad & \xleftarrow[\mu_{11}]{(0,0)} \quad \left(\frac{1}{x^2(x/y-1)}, x(\frac{x}{y}-1)\right) \xleftarrow[\mu_{12}]{(0,0)} \left(\frac{1}{x^2(x/y-1)}, x^3(\frac{x}{y}-1)^2\right) \\
& \xleftarrow[\mu_{13}]{(0,a)} \quad \left(\frac{1}{x^2(x/y-1)}, x^2(\frac{x}{y}-1)(x^3(\frac{x}{y}-1)^2-a)\right) \\
& \xleftarrow[\mu_{14}]{(0,0)} \quad \left(\frac{1}{x^2(x/y-1)}, x^4(\frac{x}{y}-1)^2(x^3(\frac{x}{y}-1)^2-a)\right)
\end{aligned}$$

Next eliminating the indeterminacy of mapping from Y_2 to Y_2, we obtain Y_3 defined by the following sequence of blow ups.

$$\begin{aligned}
(x,y) \quad & \xleftarrow[\mu_5]{\text{at } (0,\infty)} \quad \left(x, \frac{1}{xy}\right) \xleftarrow[\mu_6]{(0,0)} \left(x^2 y, \frac{1}{xy}\right) \\
& \xleftarrow[\mu_7]{(a,0)} \quad \left(xy(x^2y-a), \frac{1}{xy}\right) \xleftarrow[\mu_8]{(0,0)} \left(x^2y^2(x^2y-a), \frac{1}{xy}\right)
\end{aligned}$$

It can be shown that the mapping from Y_3 to Y_3 which is lifted from φ does not have any indeterminate points.

To show this, let us define the total and proper transforms.

Definition. Let S be the set of zero points of $\bigwedge_{i\in I} f_i(u,v)=0$, where $(u,v)\in\mathbb{C}^2$ and the $\{f_i\}_{i\in I}$ is a finite set of polynomials, and let $U_1 : (u_1,v_1)$ and $U_2 : (u_2,v_2)$ the new coordinates of blow up at the point $(u,v)=(a,b)$, i.e. $(u_1,v_1)=(u-a,(v-b)/(u-a))$, $(u_2,v_2)=((u-a)/(v-b),v-b)$. The *total transform* of S is

$$\{(u_1,v_1)\in U_1; \bigwedge_i f_i(u_1+a, u_1v_1+b)=0\} \cup \{(u_2,v_2)\in U_2; \bigwedge_i f_i(u_2v_2+a, v_2+b)=0\}$$

and the *proper transform* of S is

$$\{(u_1,v_1)\in U_1; \bigwedge_i \frac{f_i(u_1+a, u_1v_1+b)}{u_1^m}=0\} \cup \{(u_2,v_2)\in U_2; \bigwedge_i \frac{f_i(u_2v_2+a, v_2+b)}{v_2^n}=0\}$$

where m or n is the maximum integer simplifying the respective equations for u_1 or v_2 respectively.

For example, by blowing up at $(u,v)=(0,0)$, the total transform of $u=0$ is $\{(u_1,v_1)\in U_1; u_1=0\}\cup\{(u_2,v_2)\in U_2; u_2v_2=0\}$ and its proper transform is $\{(u_1,v_1)\in U_1; 1=0\}(=\phi)\cup\{(u_2,v_2)\in U_2; u_2=0\}$.

We denote the total transform of the point of the i-th blow up by E_i and denote the proper transform of the exceptional curves of the i-th blow up by

$$D_0, D_1, D_2, C_0=E_4, D_3, D_4, D_5, C_1=E_8, D_6, D_{12}, D_7, D_8, D_9, C_2=E_{14}. \tag{4}$$

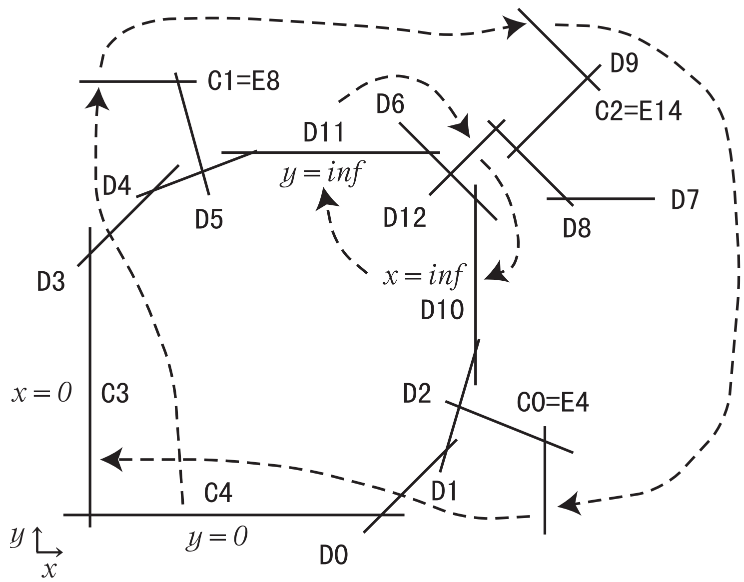


Figure 1:

Moreover we denote the proper transforms of $x = 0$, $x = \infty$, $y = 0$, $y = \infty$ as

$$C_3, D_{10}, C_4, D_{11} \tag{5}$$

(See Fig 1.)

The proper transforms of C_4, D_0, D_1, D_2 and C_0 are written as

$$\begin{aligned}
C_4 &: \quad (x, y) = (x, 0) \\
D_0 &: \quad (u_1, v_1) = (u_1, 0) \\
D_1 &: \quad (u_2, v_2) = (0, v_2) \\
D_2 &: \quad (u_3, v_3) = (0, v_3) \\
C_0 &: \quad (u_4, v_4) = (0, v_4)
\end{aligned}$$

where u_i and v_i are the new coordinate of the i-th blow up (more precisely, these express the total transforms of curves and we have to write each curve by using the coordinates of the last blow up but this makes the notation rather complicated) and therefore the relations

$$\begin{array}{ll} x = 1/(u_1 v_1), & y = v_1, \\ u_1 = u_2, & v_1 = u_2 v_2, \\ u_2 = u_3, & v_2 = u_3 v_3 + a, \\ u_3 = u_4, & v_3 = u_4 v_4 \end{array}$$

hold. The proper transforms of C_1, D_5, D_4, D_3 and C_3 are written as

$$\begin{array}{rl} C_1: & (u_8, v_8) = (u_8, 0) \\ D_5: & (u_7, v_7) = (u_7, 0) \\ D_4: & (u_6, v_6) = (u_6, 0) \\ D_3: & (u_5, v_5) = (0, v_5) \\ C_3: & (x, 1/y) = (0, 1/y) \end{array}$$

and the relations

$$\begin{array}{ll} u_8 = u_7/v_7, & v_8 = v_7, \\ u_7 = (u_6 - a)/v_6, & v_7 = v_6, \\ u_6 = u_5/v_5, & v_6 = v_5, \\ u_5 = x, & v_5 = 1/(xy) \end{array}$$

hold.

Using these relations one can calculate the images of the curves. Foe example, in the case of C_4: From the above relations and (2) we can calculate $(\overline{u_8}, \overline{v_8})$ using initial values corresponding to C_1 as

$$\begin{aligned} (\overline{u_8}, \overline{v_8})|_{(x,y)=(x,0)} &= \left((-x+y)(a+y^2(-x+y))^2, \frac{y}{a - xy^2 + y^3} \right)\Bigg|_{(x,y)=(x,0)} \\ &= (-a^2 x, 0) \end{aligned}$$

This then implies that the image of C_4 $(= \overline{C_4})$ is C_1.

Analogously, from the equation

$$\begin{aligned} & (\overline{u_7}, \overline{v_7})|_{(u_1,v_1)=(u_1,0)} \\ = & \left(\frac{(-1 + u_1 v_1^2)(a u_1 - v_1 + u_1 v_1^3)}{u_1^2}, \frac{u_1 v_1}{a u_1 - v_1 + u_1 v_1^3} \right)\Bigg|_{(u_1,v_1)=(u_1,0)} \\ = & \left(-\frac{a}{u_1}, 0 \right) \end{aligned}$$

which implies $\overline{D_0} = D_5$. In this way we can show that

$$\begin{array}{ll} & (D_0,\ D_1,\ D_2,\ D_3,\ D_4,\ D_5,\ D_6,\ D_7,\ D_8,\ D_9,\ D_{10},\ D_{11},\ D_{12},\ C_0,\ C_1,\ C_2,\ C_4) \\ \to & (D_5,\ D_4,\ D_3,\ D_7,\ D_8,\ D_9,\ D_6,\ D_0,\ D_1,\ D_2,\ D_{11},\ D_{12},\ D_{10},\ C_3,\ C_2,\ C_0,\ C_1). \end{array} \tag{6}$$

It is obvious that this mapping has an inverse (the mapping lifted from φ^{-1}). Hence we obtain the following theorem.

THEOREM **2.1** *The HV eq. (1) can be lifted to an automorphism of* $X (= Y_3)$.

3 The Picard group and symmetry

3.1 Action on the Picard group

We denote the (linear equivalent classes of) total transform of x = constant, (or y = constant) on X by H_0 (or H_1 respectively) and the (linear equivalent classes of) total transform of the point of the i-th blow up by E_i. From [9] we know that the Picard group of X, $\mathrm{Pic}(X)$, is

$$\mathrm{Pic}(X) = \mathbb{Z}H_0 + \mathbb{Z}H_1 + \mathbb{Z}E_1 + \cdots + \mathbb{Z}E_{14}$$

and that the canonical divisor of X, K_X, is

$$K_X = -2H_0 - 2H_1 + E_1 + \cdots + E_{14}$$

It is also known that the intersection form, i.e. the intersection numbers of pairs of base elements, is

$$H_i \cdot H_j = 1 - \delta_{i,j},\ E_k \cdot E_l = -\delta_{k,l},\ H_i \cdot E_k = 0 \tag{7}$$

where $\delta_{i,j}$ is 1 if $i = j$ and 0 if $i \neq j$, and the intersection numbers of any pairs of divisors are given by their linear combinations.

Remark. Let X be a rational surface. It is known that $\mathrm{Pic}(X)$, the group of isomorphism classes of invertible sheaves of X, is isomorphic to the following groups.
i)The group of linear equivalent classes of divisors on X.
ii)The group of numerically equivalent classes of divisors on X, where divisors D and D' on X are numerically equivalent if and only if for any divisors D'' on X, $D \cdot D'' = D' \cdot D''$ holds.
Hence we identify them in this paper.

The (linear equivalent classes of) prime divisors in (4), (5) as elements of $\mathrm{Pic}(X)$ are described as

$$C_0 = E_4,\ C_1 = E_8,\ C_2 = E_{14},\ C_3 = H_0 - E_5,\ C_4 = H_1 - E_1 \quad (-1 \text{ curve})$$

$$D_0 = E_1 - E_2,\ D_1 = E_2 - E_3,\ D_2 = E_3 - E_4,\ D_3 = E_5 - E_6,\ D_4 = E_6 - E_7,\ D_5 = E_7 - E_8,$$

$$D_6 = E_9 - E_{10},\ D_7 = E_{11} - E_{12},\ D_8 = E_{12} - E_{13},\ D_9 = E_{13} - E_{14} \quad (-2 \text{ curve})$$

$$D_{10} = H_0 - E_1 - E_2 - E_9,\ D_{11} = H_1 - E_5 - E_6 - E_9,\ D_{12} = E_{10} - E_{11} - E_{12} \quad (-3 \text{ curve})$$

where by n curve we mean a curve whose self-intersection number is n. See Fig.2.

The anti-canonical divisor $-K_X$ can be reduced uniquely (see appendix A) to prime divisors as

$$D_0 + 2D_1 + D_2 + D_3 + 2D_4 + D_5 + 3D_6 + D_7 + 2D_8 + D_9 + 2D_{10} + 2D_{11} + 2D_{12} \tag{8}$$

and the connection of D_i are expressed by the following diagram.

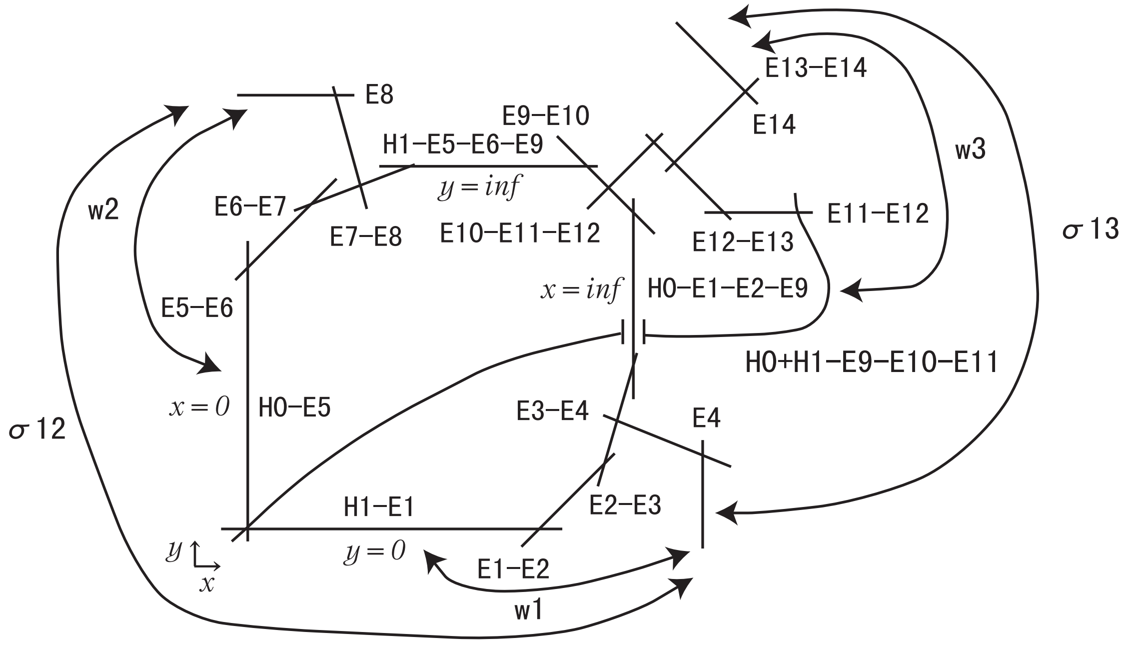
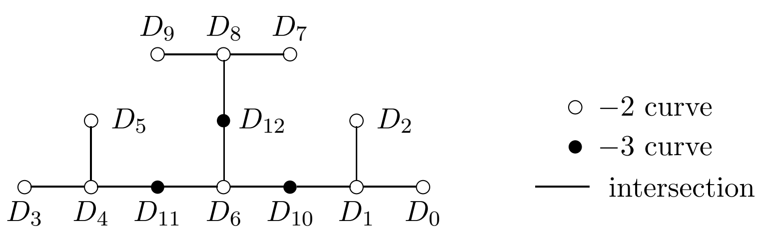


Figure 2:



(9)

The HV eq.(1) acts on curves as (6). Hence the HV eq. acts on Pic(X) as

$$\begin{pmatrix} H_0 \\ H_1, \quad E_1, \quad E_2 \\ E_3, \quad E_4, \quad E_5, \quad E_6 \\ E_7, \quad E_8, \quad E_9, \quad E_{10} \\ E_{11}, \quad E_{12}, \quad E_{13}, \quad E_{14} \end{pmatrix} \rightarrow \begin{pmatrix} 3H_0 + H_1 - E_5 - E_6 - E_7 - E_8 - E_9 - E_{10} \\ H_0, \quad H_0 - E_8, \quad H_0 - E_7 \\ H_0 - E_6, \quad H_0 - E_5, \quad E_{11}, \quad E_{12} \\ E_{13}, \quad E_{14}, \quad H_0 - E_{10}, \quad H_0 - E_9 \\ E_1, \quad E_2, \quad E_3, \quad E_4 \end{pmatrix} \tag{10}$$

(this table means $\overline{H_0} = 3H_0 + H_1 - E_5 - E_6 - E_7 - E_8 - E_9 - E_{10}$, $\overline{H_1} = H_0$, $\overline{E_1} = H_0 - E_8$

and so on) and their linear combinations.

Remark. Let θ be an isomorphism from the rational surface X to the rational surface X'. Let D be a divisor and $[D]$ its class. The class of $\theta(D)$ coincides with the class $\theta([D]) \in \mathrm{Pic}(X')$ and the action of θ on $\mathrm{Pic}(X)$ ($\cong \mathrm{Pic}(X')$) is linear.

Notice that (10) means a change of bases. Actually by fixing the basis of $\mathrm{Pic}(X)$ as $\{H_0, H_1, E_1, E_2, \cdots, E_{14}\}$, this table can be expressed by the following matrix as the action from the left hand side on the space of coefficients of basis.

$$\begin{pmatrix}
3 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix} \tag{11}$$

As we will see in Section 5, the action (10) (or (11)) provides a method to calculate the algebraic entropy of the HV eq.

3.2 Cremona isometries and the root system

Defnition. An automorphism s of $\mathrm{Pic}(X)$ is called a *Cremona isometry* if the following three properties are satisfied:
a) s preserves the intersection form in $\mathrm{Pic}(X)$;
b) s leaves K_X fixed;
c) s leaves the semigroup of effective classes of divisors invariant.

In general, if a birational mapping can be lifted to an isomorphism from X to X' by blow ups, its action on the resulting Picard group is always a Cremona isometry. We will show that the group of Cremona isometries is an extended Weyl group of hyperbolic type. In the next section we will show these Cremona isometries can be realized as Cremona transformations, i.e. birational mappings, on $\mathbb{P}^1 \times \mathbb{P}^1$.

LEMMA **3.1** *Let s be a Cremona isometry, then*
c') *s is an automorphism of the diagram(9).*

Proof. First we show that for any $i \in \{0, 1, \cdots, 12\}$ there exists $j \in \{0, \cdots, 12\}$ such that $D_i = s(D_j)$. Notice that $-K_X$ can be uniquely reduced to prime divisors in the form $-K_X = \sum_i m_i D_i$ (see (40)) and the condition b). We have $s(-K_X) = -K_X = \sum_i m_i s(D_i)$, where all $s(D_i)$ are effective divisors due to the condition c) (and moreover $D_i \cdot D_i = s(D_i) \cdot s(D_i)$). By the uniqueness of decomposition of $-K_X$, we have that for any i there exists j such that $D_i = s(D_j)$ and $m_i = m_j$. According to this fact and the condition a) we have the lemma. (Another proof is shown in [8, 13]) $\square$

Let us define $< D_i >$ and $< D_i >^\perp$ as

$$< D_i >= \sum_{i=0}^{12} \mathbb{Z} D_i$$

and

$$< D_i >^\perp = \{\alpha \in \mathrm{Pic}(X); \alpha \cdot D_i = 0 \text{ for } i = 0, 1, \cdots, 12\}.$$

LEMMA 3.2 *The Cremona isometry s leaves $< D_i >^\perp$ invariant.*

Proof. Let $\alpha \in < D_i >^\perp$. By the condition c') we have that for any $i \in \{0, \cdots, 12\}$ there exists $j \in \{0, \cdots, 12\}$ such that $s(\alpha) \cdot D_i = s(\alpha) \cdot s(D_j) = \alpha \cdot D_j = 0$. It implies $s(\alpha) \in < D_i >^\perp$. $\square$

In this case $< D_i >^\perp$ can be written as

$$< D_i >^\perp = < \alpha_i > := \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2 + \mathbb{Z}\alpha_3$$

where

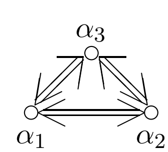
$$\begin{aligned} \alpha_1 &= 2H_1 - E_1 - E_2 - E_3 - E_4, \\ \alpha_2 &= 2H_0 - E_5 - E_6 - E_7 - E_8, \\ \alpha_3 &= 2H_0 + 2H_1 - 2E_9 - 2E_{10} - E_{11} - E_{12} - E_{13} - E_{14}. \end{aligned} \tag{12}$$

We consider $< \alpha_i >$ with the intersection form to be a root lattice with a symmetric bilinear form. Let us define the transformation $w_i (i = 1, 2, 3)$ on $< \alpha_i >$ as

$$w_i(\alpha) = \alpha - 2\frac{\alpha_i \cdot \alpha}{\alpha_i \cdot \alpha_i}\alpha_i \tag{13}$$

for $\alpha \in < \alpha_i >$. The transformation $w_i(\alpha_j)$ has the form $w_i(\alpha_j) = \alpha_j - c_{ij}\alpha_i$, where $c_{ij} = 2(\alpha_i \cdot \alpha_j)/(\alpha_i \cdot \alpha_i)$, and the matrix c_{ij} becomes a generalized Cartan matrix. Here, the generalized Cartan matrix and its Dynkin diagram are of the hyperbolic type $H_{71}^{(3)}$ [15] as follows:

$$\begin{bmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix} \quad \text{and} \tag{14}$$



Hence the group W generated by the actions w_1, w_2, w_3, is a Weyl group of hyperbolic type. The extended (including the full automorphism group of the Dynkin diagram) Weyl group, $\widetilde{W}$, is generated by

$$\{w_1, w_2, w_3, \sigma_{12}, \sigma_{13}\} \tag{15}$$

and the fundamental relations:

$$\begin{gathered} w_i^2 = \sigma_{1j}^2 = 1, \quad (\sigma_{12}\sigma_{13})^3 = 1, \\ \sigma_{12}w_1 = w_2\sigma_{12}, \quad \sigma_{12}w_2 = w_1\sigma_{12}, \quad \sigma_{12}w_3 = w_3\sigma_{12} \\ \sigma_{13}w_1 = w_3\sigma_{13}, \quad \sigma_{13}w_2 = w_2\sigma_{13}, \quad \sigma_{13}w_3 = w_1\sigma_{13} \end{gathered} \tag{16}$$

where the action of σ_{12} or σ_{13} on $< \alpha_i >$ is defined by the exchange of indices of α_i; the action of σ_{1j} and w_k on α_i can be summarized as follows:

$\sigma_j(\alpha_i)$ and $w_k(\alpha_i)$

	σ_{12}	σ_{13}	w_1	w_2	w_3
$\alpha_1 \mapsto$	α_2	α_3	$-\alpha_1$	$\alpha_1 + 2\alpha_2$	$\alpha_1 + 2\alpha_3$
$\alpha_2 \mapsto$	α_1	α_2	$\alpha_2 + 2\alpha_1$	$-\alpha_2$	$\alpha_2 + 2\alpha_3$
$\alpha_3 \mapsto$	α_3	α_1	$\alpha_3 + 2\alpha_1$	$\alpha_3 + 2\alpha_2$	$-\alpha_3$

. (17)

Moreover, by the following property we have the fact that the group of Cremona isometries is included in $\pm\widetilde{W}$.

PROPOSITION 3.3 ([16] §5.10) *If the generalized Cartan Matrix c_{ij} is a symmetric matrix of finite, affine, or hyperbolic type, then the group of all automorphisms of $< \alpha_i >$ preserving the bilinear form is $\pm\widetilde{W}$.*

Remark. If s is a Cremona isometry, then $-s$ can not satisfy the condition c).

Next we consider uniqueness for the extension of action of $\widetilde{W}$ to the action on Pic(X).

LEMMA 3.4 *Let s and s' be Cremona isometries such that the action of s is identical to the action of s' on $< \alpha_i >$, then $s = s'$ as Cremona isometries, i.e. s is identical to s' as an automorphism of Pic(X).*

Proof. Let s and s' be Cremona isometries such that the actions of s is identical to the action of s' on $< \alpha_i >$. The actions of $s' \circ s^{-1}$ on $< \alpha_i >$ is the identity.

We investigate where the exceptional divisor E_4 is moved by the action of $s' \circ s^{-1}$. In $\{D_i\,; i = 0, \cdots, 12\}$, only D_2 has an intersection with E_4. By the condition c'), $s' \circ s^{-1}(D_2)$ is D_0, D_2, D_3, D_5, D_7 or D_9 and only $s' \circ s^{-1}(D_2)$ has an intersection with $s' \circ s^{-1}(E_4)$ in $\{s' \circ s^{-1}(D_i);\ i = 0, \cdots, 12\}$.

i) Assume $s' \circ s^{-1}(D_2) = D_2$. $s' \circ s^{-1}(E_4)$ has an intersection only with D_2 in $\{D_i\,; i = 0, \cdots, 12\}$ (this condition on the coefficients of basis of Pic(X) can be considered to be a system of linear equations of order 13). Then we have the general solution with three integers z_1, z_2, z_3:

$$s' \circ s^{-1}(E_4) = E_4 + z_1\alpha_1 + z_2\alpha_2 + z_3\alpha_3.$$

Multiplying this equation by $s' \circ s^{-1}(\alpha_i) = \alpha_i$, we have the system of linear equations:

$$\left\{ \begin{array}{rcl} 1 - 4z_1 + 4z_2 + 4z_3 & = & 1 \\ 0 + 4z_1 - 4z_2 + 4z_3 & = & 0 \\ 0 + 4z_1 + 4z_2 - 4z_3 & = & 0 \end{array} \right. .$$

It implies $s' \circ s^{-1}(E_4) = E_4$.

ii) Assume $s' \circ s^{-1}(D_2) = D_0$. We have $s' \circ s^{-1}(E_4) = (H_1 - E_1) + z_1\alpha_1 + z_2\alpha_2 + z_3\alpha_3$. Multiplying this equation by $s' \circ s^{-1}(\alpha_i) = \alpha_i$, one has that this equation does not have integer solutions.

iii) The other cases. $s' \circ s^{-1}(D_2) = D_3, D_5, D_7$ or D_9 implies $s' \circ s^{-1}(E_4) = L + z_1\alpha_1 + z_2\alpha_2 + z_3\alpha_3$, where $L = H_0 - E_5, E_8, H_0 + H_1 - E_9 - E_{10} - E_{11}$ or E_{14} respectively. This implies that this equation does not have integer solutions.

According to i),ii) and iii) $s' \circ s^{-1}(E_4) = E_4$ and $s' \circ s^{-1}(D_2) = D_2$.

Analogously we have $s' \circ s^{-1}(H_1 - E_1) = H_1 - E_1$ and $s' \circ s^{-1}(D_0) = D_0$ and so on. Due to this fact and the condition c'), $s' \circ s^{-1}$ must be the identity as an Cremona isometry. This implies the lemma. $\square$

Next we consider the extension of actions of elements of $\widetilde{W}$ on $< \alpha_i >$ to the actions on Pic(X). Let us define $\alpha_{i,j}$ $(i = 1, 2, 3\ j = 1, 2)$ as

$$\begin{array}{ll} \alpha_{1,1} = H_1 - E_1 - E_4, & \alpha_{1,2} = H_1 - E_2 - E_3, \\ \alpha_{2,1} = H_0 - E_5 - E_7, & \alpha_{2,2} = H_0 - E_6 - E_7, \\ \alpha_{3,1} = H_0 + H_1 - E_9 - E_{10} - E_{11} - E_{14}, & \alpha_{3,2} = H_0 + H_1 - E_9 - E_{10} - E_{12} - E_{13} \end{array} \tag{18}$$

and define the action of $\alpha_{i,j}$ $(i = 1, 2, 3\ j = 1, 2)$ on $\alpha \in < \alpha_i >$ as

$$w_{i,j}(\alpha) := \alpha - 2\frac{\alpha_{i,j} \cdot \alpha}{\alpha_{i,j} \cdot \alpha_{i,j}}\alpha_{i,j} \quad .$$

It is easy to see that $w_i(\alpha) = w_{i,1} \circ w_{i,2}(\alpha) = w_{i,2} \circ w_{i,1}(\alpha)$. We define the action of w_i on $D \in$Pic(X) as $w_i(D) := w_{i,1} \circ w_{i,2}(D) = w_{i,2} \circ w_{i,1}(D)$. These actions are explicitly written as follows (See Fig.2). (For brevity we did not write the invariant elements under each action.)

$$\begin{array}{lccc} w_1: & \begin{pmatrix} H_0, \\ E_1, \ E_2, \ E_3, \ E_4 \end{pmatrix} & \to & \begin{pmatrix} H_0 + 2H_1 - E_1 - E_2 - E_3 - E_4 \\ H_1 - E_4, \ H_1 - E_3, \ H_1 - E_2, \ H_1 - E_1 \end{pmatrix} \\ w_2: & \begin{pmatrix} H_1, \\ E_5, \ E_6, \ E_7, \ E_8 \end{pmatrix} & \to & \begin{pmatrix} 2H_0 + H_1 - E_5 - E_6 - E_7 - E_8 \\ H_0 - E_8, \ H_0 - E_7, \ H_0 - E_6, \ H_0 - E_5 \end{pmatrix} \\ w_3: & \begin{pmatrix} H_0, \ H_1, \ E_9, \ E_{10} \\ E_{11}, \ E_{12}, \ E_{13}, \ E_{14} \end{pmatrix} & \to & \begin{pmatrix} H_0 + \alpha_3, \ H_1 + \alpha_3, \ E_9 + \alpha_3, \ E_{10} + \alpha_3 \\ E_{11} + \alpha_{3,1}, \ E_{12} + \alpha_{3,2}, \ E_{13} + \alpha_{3,2}, \ E_{14} + \alpha_{3,1} \end{pmatrix} \end{array} \tag{19}$$

We define the action of σ_{12} and σ_{13} on $\mathrm{Pic}(X)$ as follows.

$$\begin{array}{ll}\sigma_{12}: & \begin{pmatrix} H_0, & H_1, & E_1, & E_2, & E_3 \\ E_4, & E_5, & E_6, & E_7, & E_8 \end{pmatrix} \to \begin{pmatrix} H_1, & H_0, & E_5, & E_6, & E_7 \\ E_8, & E_1, & E_2, & E_3, & E_4 \end{pmatrix} \\ \sigma_{13}: & \begin{pmatrix} H_1, \; E_1, \; E_2 \\ E_3, \; E_4, \; E_9, \; E_{10} \\ E_{11}, \; E_{12} \; E_{13}, \; E_{14} \end{pmatrix} \to \begin{pmatrix} H_0+H_1-E_9-E_{10}, \; E_{11}, \; E_{12} \\ E_{13}, \; E_{14}, \; H_0-E_{10}, \; H_0-E_9 \\ E_1, \; E_2, \; E_3, \; E_4 \end{pmatrix}\end{array} \qquad (20)$$

By direct calculation, it is easy to check that each w_i (or σ_{1i}) expressed by (19) (or (20) resp.) acts on $< \alpha_i >$ as (13) (or as the exchanges of indices of α_i resp.) and that they satisfy the fundamental relations (16) (the later property is of course guaranteed by the uniqueness of extension of $\widetilde{W}$). Moreover it is also easy to check that the actions of all elements of $\widetilde{W}$ on $\mathrm{Pic}(X)$ satisfy the conditions a),b) and c').

THEOREM 3.5 *The group of Cremona isometries of X is isomorphic to $\widetilde{W}$, where $\widetilde{W}$ is generated by $\{w_1, w_2, w_3, \sigma_{12}, \sigma_{13}\}$ and the fundamental relations (16). The actions of elements of $\widetilde{W}$ on Pic(X) are given by (19) and (20) and their composition.*

To show this theorem, it is enough to show that (19) and (20) satisfy the condition c). For this purpose, it is enough to realize them as isomorphisms from X to X', where X and X' have the same semigroup of classes of effective divisors. We show this fact in the next subsection.

From (19) and (20) it is straightforward to show that the action of the HV eq. on $\mathrm{Pic}(X)$ is identical to the action of $w_2 \circ \sigma_{13} \circ \sigma_{12}$.

COROLLARY 3.6 *There does not exist any Cremona isometry of X whose action on Pic(X) commutes with the action of the HV eq. except $(w_2 \circ \sigma_{13} \circ \sigma_{12})^m$, where $m \in \mathbb{Z}$.*

Proof. In this proof we denote σ_{12} or σ_{13} by σ_2 or σ_3 respectively and omit the symbol of composition $\circ$. Each element of $\widetilde{W}$ can be uniquely written in the form

$$w_{i_1} w_{i_2} \cdots w_{i_n} s$$

where all indices of w (or σ) are considered in Mod 3 (or 2 resp.) and $i_l \neq i_{l+1}$ and $s \in \{1, \sigma_j, \sigma_j\sigma_{j+1}, \sigma_j\sigma_{j+1}\sigma_j\}$. Assume $g = w_{i_1} w_{i_2} \cdots w_{i_n} s$ commutes with $w_2\sigma_3\sigma_2$.

i) The case of $s = 1$. According to the relation

$$w_2\sigma_3\sigma_2 w_{i_1} w_{i_2} \cdots w_{i_n} = w_{i_1} w_{i_2} \cdots w_{i_n} w_2\sigma_3\sigma_2,$$

we have the relation

$$w_2 w_{i_1+1} w_{i_2+1} \cdots w_{i_n+1} \sigma_3\sigma_2 = w_{i_1} w_{i_2} \cdots w_{i_n} w_2\sigma_3\sigma_2.$$

It implies $i_1 \equiv 2, i_2 \equiv 3, \cdots, i_n \equiv n+1, 2 \equiv n+2$ and therefore there exists the integer m such that $n = 3m$. On the other hand $(w_2\sigma_3\sigma_2)^{3m} = w_2 w_3 \cdots w_{n+1}$. It implies $g = (w_2\sigma_3\sigma_2)^{3m}$.

ii) The case of $s = \sigma_3\sigma_2$ or $\sigma_2\sigma_3$. Similar to the case i), n must be $n = 3m + 1$ or $n = 3m + 2$ respectively and g becomes $(w_2\sigma_3\sigma_2)^n$.

iii) The case of $s = \sigma_j$. Suppose $j = 2$. According to the relation

$$w_2\sigma_3\sigma_2 w_{i_1} w_{i_2} \cdots w_{i_n}\sigma_2 = w_{i_1} w_{i_2} \cdots w_{i_n}\sigma_2 w_2\sigma_3\sigma_2,$$

we have the relation

$$w_2 w_{i_1+1} w_{i_2+1} \cdots w_{i_n+1}\sigma_3 = w_{i_1} w_{i_2} \cdots w_{i_n} w_1\sigma_2\sigma_3\sigma_2.$$

It implies $\sigma_3 = \sigma_2\sigma_3\sigma_2$ but this is a contradiction. Similarly $s = \sigma_3$ is impossible.

v) The case $s = \sigma_j\sigma_{j+1}\sigma_j$. Suppose $j = 2$. Similar to the case iii), we have the relation

$$w_2 w_{i_1+1} w_{i_2+1} \cdots w_{i_n+1}\sigma_2 = w_{i_1} w_{i_2} \cdots w_{i_n} w_3\sigma_2\sigma_3\sigma_2\sigma_3\sigma_2.$$

It implies $\sigma_2 = \sigma_2\sigma_3\sigma_2\sigma_3\sigma_2$ and hence $1 = \sigma_3\sigma_2\sigma_3\sigma_2$ which leads to a contradiction. Similarly it can be shown that $s = \sigma_3\sigma_2\sigma_3$ is impossible. $\square$

4 The inverse problem

A birational mapping is called a Cremona transformation. One method for obtaining a Cremona transformation such that its action on $\mathrm{Pic}(X)$ is a Cremona isometry is to interchange the blow down structures, i.e. to interchange the procedure of blow downs. Following this method, we construct the Cremona transformations on $\mathbb{P}^1 \times \mathbb{P}^1$ which yield the extended Weyl group (15), (16) and thereby recover the HV eq. from its action on $\mathrm{Pic}(X)$.

An element of $\widetilde{W}$ is an automorphism of $\mathrm{Pic}(X)$ but does not have to be an automorphism of X itself, i.e. the blow up points can be changed with a transformation satisfying the condition a),b) and c) in Section 3.2. In order to do this, one has to consider not only autonomous but also non-autonomous mappings.

By $a_0, a_1, a_2, a_3, a_4, a_5, a_6$ or a_7 we denote the point of the $10, 3, 4, 7, 8, 11, 13, 14$-th blow up or the corresponding coordinates and we call them "the parameters". In short these points can be expressed as follows:

$$
\begin{aligned}
&\left(\frac{1}{xy}, xy^2\right) = (0, a_1), \quad \left(\frac{1}{xy}, xy(xy^2 - a_1)\right) = (0, a_2),\\
&\left(x^2y, \frac{1}{xy}\right) = (a_3, 0)), \quad \left(xy(x^2y - a_3), \frac{1}{xy}\right) = (a_4, 0),\\
&\left(\frac{1}{x}, \frac{x}{y}\right) = (0, a_0)), \quad \text{where we normalize } a_0 \text{ to be } a_0 = 1,\\
&\left(\frac{1}{x}, x(\frac{x}{y} - 1)\right) = (0, a_5), \quad \left(\frac{1}{x(x(\frac{x}{y} - 1) - a_5)}, x(x(\frac{x}{y} - 1) - a_5)^2\right) = (0, a_6),\\
&\left(\frac{1}{x(x(\frac{x}{y} - 1) - a_5)}, x(x(\frac{x}{y} - 1) - a_5)\left\{x(x(\frac{x}{y} - 1) - a_5)^2 - a_6\right\}\right) = (0, a_7),
\end{aligned}
$$

where $a_i \in \mathbb{C}$ and a_1, a_3, a_6 are nonzero.

The point of the $2, 6, 12$-th blow up is determined by intersection numbers. Moreover the point of the $1, 5, 9, 10, 11$-th blow up can be fixed by acting with a suitable automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$, i.e. a Möbius transformation of each coordinate combination with an exchange of the coordinates. We call this operation "normalization". It can also be seen that we can normalize $a_5 = 1$ except the case $a_5 = 0$.

In this section we consider a realization of the generating elements of $\widetilde{W}$ as Cremona transformations which can be lifted to isomorphisms from X to $\overline{X}$, where $\overline{X}$ is the same rational surface as X except for a difference in parameters.

First we realize the action of w_2 as a Cremona transformation on $\mathbb{P}^1 \times \mathbb{P}^1$

4.1 The calculation of interchanging the blow down structure

In the following we shall present a scheme in which the blow down structure is changed. This method is based on the following fact:

By F_n we denote the n-th Hirzebruch surface with the coordinate system

$$(\bigcirc, \triangle) \cup (\bigcirc, \frac{1}{\triangle}) \cup (\frac{1}{\bigcirc}, \bigcirc^n \triangle) \cup (\frac{1}{\bigcirc}, \frac{1}{\bigcirc^n \triangle}). \tag{21}$$

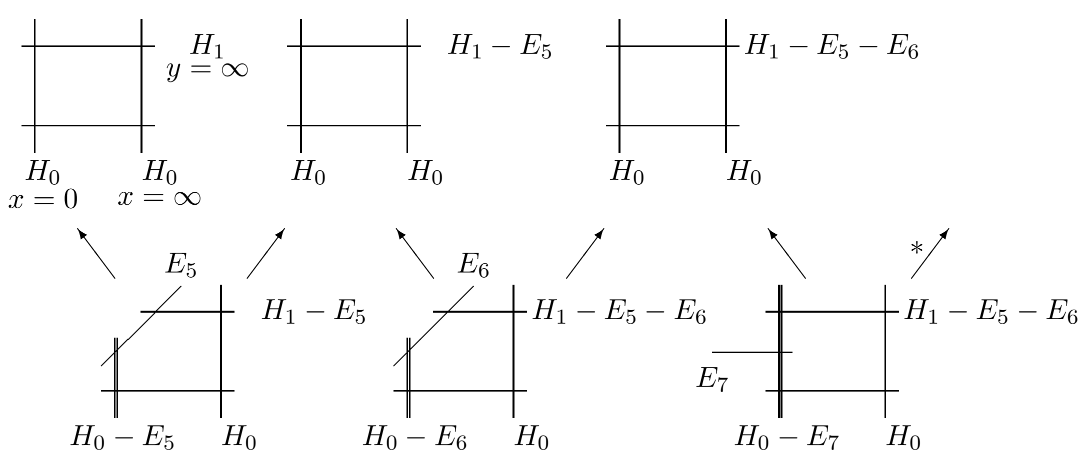
Blowing up the n-th Hirzebruch surface at the point $(1/\bigcirc, \bigcirc^n\triangle) = (0,0)$ and blowing down along the line $1/\bigcirc = 1/(\bigcirc^{n+1}\triangle) = 0$, we obtain the $n+1$-th Hirzebruch surface as follows:

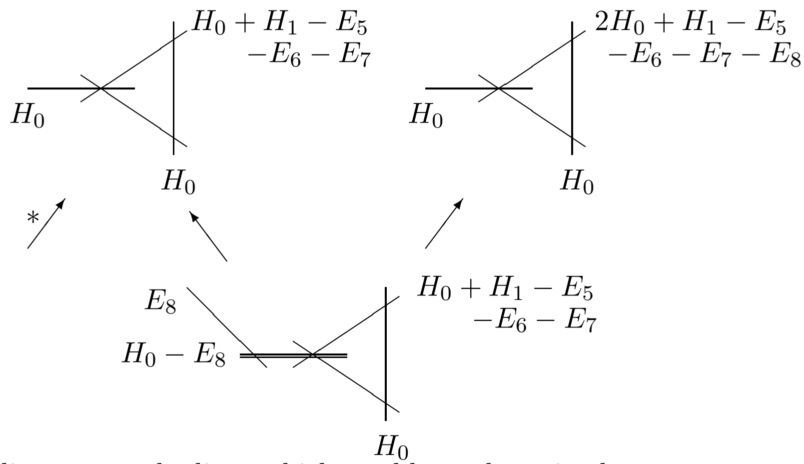
$$\begin{array}{lllll}
 & (\bigcirc, \triangle) & \cup(\bigcirc, \frac{1}{\triangle}) & \cup(\frac{1}{\bigcirc}, \bigcirc^n\triangle) & \cup(\frac{1}{\bigcirc}, \frac{1}{\bigcirc^n\triangle}) \\
\xleftarrow{\text{up}} & (\bigcirc, \triangle) & \cup(\bigcirc, \frac{1}{\triangle}) & \cup(\frac{1}{\bigcirc}, \bigcirc^{n+1}\triangle) \cup (\frac{1}{\bigcirc^{n+1}\triangle}, \bigcirc^n\triangle) & \cup(\frac{1}{\bigcirc}, \frac{1}{\bigcirc^n\triangle}) \\
\xrightarrow{\text{down}} & (\bigcirc, \triangle) & \cup(\bigcirc, \frac{1}{\triangle}) & \cup(\frac{1}{\bigcirc}, \bigcirc^{n+1}\triangle) & \cup(\frac{1}{\bigcirc}, \frac{1}{\bigcirc^{n+1}\triangle}).
\end{array}$$

On the other hand, blowing up the n-th Hirzebruch surface at the point $(1/\bigcirc, 1/(\bigcirc^n\triangle)) = (0,0)$ and blowing down along the line $1/\bigcirc = \bigcirc^{n-1}\triangle = 0$, we obtain the $n-1$-th Hirzebruch surface as follows:

$$\begin{array}{lllll}
 & (\bigcirc, \triangle) & \cup(\bigcirc, \frac{1}{\triangle}) & \cup(\frac{1}{\bigcirc}, \bigcirc^n\triangle) & \cup(\frac{1}{\bigcirc}, \frac{1}{\bigcirc^n\triangle}) \\
\xleftarrow{\text{up}} & (\bigcirc, \triangle) & \cup(\bigcirc, \frac{1}{\triangle}) & \cup(\frac{1}{\bigcirc}, \bigcirc^n\triangle) & \cup(\frac{1}{\bigcirc}, \frac{1}{\bigcirc^{n-1}\triangle}) \cup (\bigcirc^{n-1}\triangle, \frac{1}{\bigcirc^n\triangle}) \\
\xrightarrow{\text{down}} & (\bigcirc, \triangle) & \cup(\bigcirc, \frac{1}{\triangle}) & \cup(\frac{1}{\bigcirc}, \bigcirc^{n-1}\triangle) & \cup(\frac{1}{\bigcirc}, \frac{1}{\bigcirc^{n-1}\triangle}).
\end{array}$$

The Next figure shows the order of the blow ups and the blow downs to obtain the Cremona transformation corresponding to w_2. (This table has to be read from the left to the right.)





where double lines mean the lines which are blown down in the next steps.

The calculation of changing the blow down structure is as follows.

$$
\begin{aligned}
& (x,y)\cup(x,\frac{1}{y})\cup(\frac{1}{x},y)\cup(\frac{1}{x},\frac{1}{y}) \quad = \mathbb{P}^1\times\mathbb{P}^1 \\
\leftarrow\ & (x,y)\cup(x,\frac{1}{xy})\cup(xy,\frac{1}{y})\cup(\frac{1}{x},y)\cup(\frac{1}{x},\frac{1}{y}) \\
\rightarrow\ & (x,xy)\cup(x,\frac{1}{xy})\cup(\frac{1}{x},y)\cup(\frac{1}{x},\frac{1}{y}) \quad = F_1 \\
\leftarrow\ & (x,xy)\cup(x,\frac{1}{x^2y})\cup(x^2y,\frac{1}{xy})\cup(\frac{1}{x},y)\cup(\frac{1}{x},\frac{1}{y}) \\
\rightarrow\ & (x,x^2y)\cup(x,\frac{1}{x^2y})\cup(\frac{1}{x},y)\cup(\frac{1}{x},\frac{1}{y}) \quad = F_2 \\
\sim\ & \quad (x,x^2y-a_3)\cup(x,\frac{1}{x^2y-a_3})\cup(\frac{1}{x},\frac{x^2y-a_3}{x^2})\cup(\frac{1}{x},\frac{x^2}{x^2y-a_3}) \quad = F_2 \\
\leftarrow\ & (x,\frac{x^2y-a_3}{x})\cup(\frac{x}{x^2y-a_3},x^2y-a_3)\cup(x,\frac{1}{x^2y-a_3})\cup(\frac{1}{x},\frac{x^2y-a_3}{x^2})\cup(\frac{1}{x},\frac{x^2}{x^2y-a_3}) \\
\rightarrow\ & (x,\frac{x^2y-a_3}{x})\cup(x,\frac{x}{x^2y-a_3})\cup(\frac{1}{x},\frac{x^2y-a_3}{x^2})\cup(\frac{1}{x},\frac{x^2}{x^2y-a_3}) \quad = F_1 \\
\sim\ & \quad (x,\frac{a_3(x^2y-a_3)-a_4x}{a_3x})\cup(x,\frac{a_3x}{a_3(x^2y-a_3)-a_4x})\cup\cdots \quad = F_1 \\
\leftarrow\ & (x,\frac{a_3(x^2y-a_3)-a_4x}{a_3x^2})\cup(\frac{a_3x^2}{(a_3(x^2y-a_3)-a_4x},\frac{a_3(x^2y-a_3)-a_4x}{a_3x}) \\
& \qquad\qquad \cup(x,\frac{a_3x}{a_3(x^2y-a_3)-a_4x})\cup\cdots \\
\rightarrow\ & (x,\frac{a_3(x^2y-a_3)-a_4x}{a_3x^2})\cup(x,\frac{a_3x^2}{a_3(x^2y-a_3)-a_4x})\cup\cdots \quad = \mathbb{P}^1\times\mathbb{P}^1
\end{aligned}
$$

where $\cdots$ is

$$
(\frac{1}{x},\frac{a_3(x^2y-a_3)-a_4x}{a_3x^2})\cup(\frac{1}{x},\frac{a_3x^2}{a_3(x^2y-a_3)-a_4x})
$$

and $\sim$ means an automorphism of the Hirzebruch surface and is determined as the point of blow up in (21) is moved to the origin.

Writing

$$w_2' : (x,y) \mapsto (x, y - \frac{a_3}{x^2} - \frac{a_4}{a_3 x})$$

we obtain $w_2 = t \circ w_2'$ where t is an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$. By taking a suitable t, we can normalize w_2 to get the required result.

4.2 Normalization and the action on the space of parameters

First we determine the automorphism for normalization t. By (19), w_2 does not move the points $(x,y) = (\infty, 0), (\infty, \infty)$. According to the fact: $w_2' : (\infty, 0) \mapsto (\infty, 0), (\infty, \infty) \mapsto (\infty, \infty)$, t is reduced to the mapping $t : (x,y) \mapsto (c_1 x + c_2, c_3 y)$, where $c_1, c_2, c_3 \in \mathbb{C}$ are nonzero constants.

Similarly, w_2 moves the point a_3 to the proper transform of the point of the 6-th blow up. We denote this fact as

$$(\overline{u_6}, \overline{v_6})|_{(u_6,v_6)=(a_3,0)} = (0,0),$$

where (u_n, v_n) is the coordinate of the n-th blow up. On the other hand,

$$\begin{aligned}
& (\overline{u_6}, \overline{v_6})|_{(u_6,v_6)=(a_3,0)} \\
= \; & \left(\bar{x}^2 \bar{y}, \frac{1}{\bar{x}\bar{y}}\right)\Big|_{(u_6,v_6)=(a_3,0)} \\
= \; & \left(x^2 (y - \frac{a_3}{x^2} - \frac{a_4}{a_3 x}), \frac{a_3 x}{a_3(x^2 y - a_3) - a_4 x}\right)\Big|_{(u_6,v_6)=(a_3,0)} \\
= \; & \left(-a_3 + u_6 - \frac{a_4 u_6 v_6}{a_3}, -\frac{a_3 u_6 v_6}{a_3^2 - a_3 u_6 + a_4 u_6 v_6}\right)\Big|_{(u_6,v_6)=(a_3,0)} \\
= \; & (0,0)
\end{aligned}$$

holds. Hence t does not move the point $(x,y) = (0,\infty)$ and therefore $c_2 = 0$.

Similarly, since w_2 does not move the points of the 10-th and the 11-th blow ups, we have $c_1 = c_3 = 1$ (moreover we can normalize a_5 to be $a_5 = 1$ or $a_5 = 0$ by taking a suitable value of $c_1 = c_3$). Hence t has to be the identity.

Next we calculate how the parameters $a_1, a_2, a_3, a_4, a_6, a_7$ are changed by the action of w_2. Notice that w_2 is an isomorphism from X to X', where w_2 satisfy the conditions a),b) and c) in Section 3.2. and therefore X and X' have the same sequence of blow ups except their parameters. By $\overline{a_i}$ we denote the parameter of the i-th blow up of X'.

Since the action of w_2 moves the points of blow ups as follows

$$\begin{aligned}
(u_2, v_2) = (0, a_1) & \mapsto (\overline{u_2}, \overline{v_2}) = (0, \overline{a_1}) \\
(u_3, v_3) = (0, a_2) & \mapsto (\overline{u_3}, \overline{v_3}) = (0, \overline{a_2}) \\
(u_6, v_6) = (0, 0) & \mapsto (\overline{u_6}, \overline{v_6}) = (\overline{a_3}, 0) \\
(xy, 1/y) = (0, 0) & \mapsto (\overline{u_7}, \overline{v_7}) = (\overline{a_4}, 0) \\
(u_{12}, v_{12}) = (0, a_6) & \mapsto (\overline{u_{12}}, \overline{v_{12}}) = (0, \overline{a_6}) \\
(u_{13}, v_{13}) = (0, a_7) & \mapsto (\overline{u_{13}}, \overline{v_{13}}) = (0, \overline{a_7}),
\end{aligned}$$

$\overline{a_i}$ can be calculated. For example $\overline{a_1}$ is calculated as follows

$$\begin{aligned}
(0, \overline{a_1}) &= (\overline{u_2}, \overline{v_2})|_{(u_2,v_2)=(0,a_1)} \\
&= \left(\frac{1}{\bar{x}\bar{y}}, \bar{x}\bar{y}^2)\right|_{(u_2,v_2)=(0,a_1)} \\
&= \left.\left(\frac{a_3u_2}{a_3 - a_4u_2 - a_3^2u_2^3v_2}, \frac{v_2(-a_3 + a_4u_2 + a_3^2u_2^3v_2)^2}{a_3^2}\right)\right|_{(u_2,v_2)=(0,a_1)} \\
&= (0, a_1),
\end{aligned}$$

and therefore $\overline{a_1} = a_1$.

Similarly we can calculate $\overline{a_2}, \overline{a_3}, \overline{a_4}$ as follows:

$$\begin{aligned}
(0, \overline{a_2}) &= \left(\frac{1}{\bar{x}\bar{y}}, \bar{x}\bar{y}(\bar{x}\bar{y}^2 - \overline{a_1})\right)\Big|_{(u_3,v_3)=(0,a_2)} \\
&= (0, a_2 - \frac{a_1a_4}{a_3}),
\end{aligned}$$

$$\begin{aligned}
(\overline{a_3}, 0) &= \left(\bar{x}^2\bar{y}, \frac{1}{\bar{x}\bar{y}}\right)\Big|_{(u_6,v_6)=(0,0)} \\
&= (-a_3, 0),
\end{aligned}$$

$$\begin{aligned}
(\overline{a_4}, 0) &= \left(\bar{x}\bar{y}(\bar{x}^2\bar{y} - \overline{a_3}), \frac{1}{\bar{x}\bar{y}}\right)\Big|_{(xy,1/y)=(0,0)} \\
&= (a_4, 0).
\end{aligned}$$

Consequently w_2 changes the parameters a_i as

$$\begin{array}{llll}
\overline{a_1} = a_1, & \overline{a_2} = a_2 - 2a_1a_4/a_3, & \overline{a_3} = -a_3, & \overline{a_4} = a_4, \\
\overline{a_5} = a_5, & \overline{a_6} = a_6, & \overline{a_7} = a_7 + 2a_4a_6/a_3 & .
\end{array}$$

We write the action of w_2 on $\mathbb{P}^1 \times \mathbb{P}^1$ and the space of parameters together as

$$\begin{aligned}
w_2 : \quad & (x, y; a_1, a_2, a_3, a_4, a_5, a_6, a_7) \\
\mapsto \quad & (\overline{x}, \overline{y}\ ;\ \overline{a_1}, \overline{a_2}, \overline{a_3}, \overline{a_4}, \overline{a_5}, \overline{a_6}, \overline{a_7}) \\
= \quad & \left(x, y - \frac{a_3}{x^2} - \frac{a_4}{a_3x}\ ;\ a_1, a_2 - \frac{2a_1a_4}{a_3}, -a_3, a_4, a_5, a_6, a_7 + \frac{2a_4a_6}{a_3}\right). \qquad (22)
\end{aligned}$$

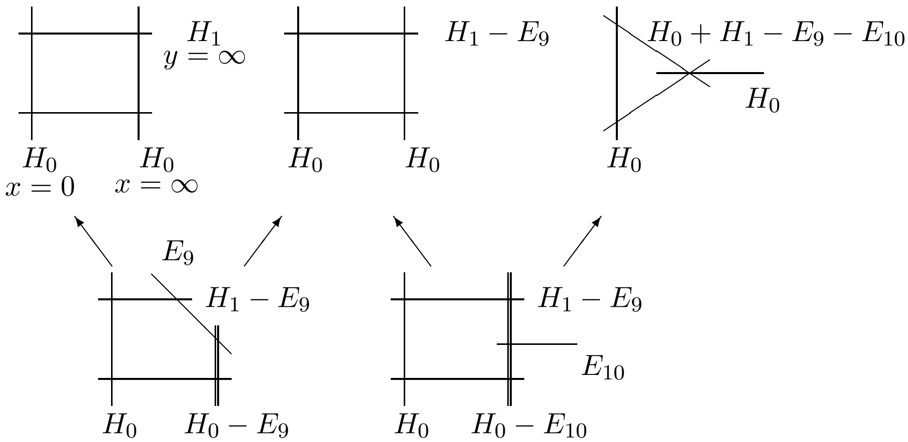
Here, in the calculation of the next iteration step we have to use $\overline{a_3} = -a_3$ instead of a_3 and so on.

As was remarked before the mapping w_2 is of order 2 as an element of an extended Weyl group and can be lifted to an isomorphism from X to $\overline{X}$.

4.3 The actions of other elements

Next we calculate the action of σ_{13} from X to $\overline{X}$.

The following figure shows the order of the blow ups and the blow downs.



Its calculation is as follows.

$$\begin{aligned}
& (x,y)\cup(x,\frac{1}{y})\cup(\frac{1}{x},y)\cup(\frac{1}{x},\frac{1}{y}) = \mathbb{P}^1\times\mathbb{P}^1 \\
\leftarrow\quad & (x,y)\cup(x,\frac{1}{y})\cup(\frac{1}{x},y)\cup(\frac{1}{x},\frac{x}{y})\cup(\frac{y}{x},\frac{1}{y}) \\
\rightarrow\quad & (x,y)\cup(x,\frac{1}{y})\cup(\frac{1}{x},\frac{y}{x})\cup(\frac{1}{x},\frac{x}{y}) = F_1 \\
\sim\quad & (x,y-x)\cup(x,\frac{1}{y-x})\cup(\frac{1}{x},\frac{y-x}{x})\cup(\frac{1}{x},\frac{x}{y-x}) = F_1 \\
\leftarrow\quad & (x,y-x)\cup(x,\frac{1}{y-x})\cup(\frac{1}{x},y-x)\cup(\frac{1}{y-x},\frac{y-x}{x})\cup(\frac{1}{x},\frac{x}{y-x}) \\
\rightarrow\quad & (x,y-x)\cup(x,\frac{1}{y-x})\cup(\frac{1}{x},y-x)\cup(\frac{1}{x},\frac{1}{y-x}) = \mathbb{P}^1\times\mathbb{P}^1
\end{aligned}$$

Similar to the case of w_2, we have the action of σ_{13} on X and the space of parameters as follows

$$\begin{aligned}
\sigma_{13}:\quad & (x,y;a_1,a_2,a_3,a_4,a_5,a_6,a_7) \\
\mapsto\quad & \big(x,\ x-y-a_5\ ;\ a_6, a_7-2a_5^2a_6,\ \ -a_3,a_4,a_5,a_1,a_2+2a_1a_5^2\big)\,.
\end{aligned} \tag{23}$$

Similarly the action of σ_{12} on X and the space of parameters is

$$\begin{aligned}
\sigma_{12}:\quad & (x,y;a_1,a_2,a_3,a_4,a_5,a_6,a_7) \\
\mapsto\quad & \big(-y,\ \ -x\ ;\ \ -a_3,-a_4,\ \ -a_1,-a_2,a_5,-a_6,a_7-4a_5^2a_6\big)\,.
\end{aligned} \tag{24}$$

The action of w_1 or w_3 is determined by the relation $w_1=\sigma_{12}\circ w_2\circ\sigma_{12}$ or $w_3=\sigma_{13}\circ w_1\circ\sigma_{13}$ respectively as follows

$$\begin{aligned}
w_1:\quad & (x,y;a_1,a_2,a_3,a_4,a_5,a_6,a_7) \\
\mapsto\quad & \left(x-\frac{a_1}{y^2}-\frac{a_2}{a_1y},y\ ;\ \ -a_1,a_2,\ a_3,a_4-\frac{2a_2a_3}{a_1},a_5,a_6,a_7+\frac{2a_2a_6}{a_1}\right).
\end{aligned} \tag{25}$$

$$
\begin{aligned}
&w_3 : (x, y; a_1, a_2, a_3, a_4, a_5, a_6, a_7)\\
&\mapsto \Bigg(x - \frac{a_6}{(x-y-a_5)^2} - \frac{a_7 - 2a_5^2 a_6}{a_6(x-y-a_5)},\ y - \frac{a_6}{(x-y-a_5)^2} - \frac{a_7 - 2a_5^2 a_6}{a_6(x-y-a_5)}\ ;\\
&a_1, a_2 + \frac{2a_1(a_7 - 2a_5^2 a_6)}{a_6}, a_3, a_4 + \frac{2a_3(a_7 - 2a_5^2 a_6)}{a_6}, a_5, -a_6, a_7 - 4a_5^2 a_6 \Bigg). \qquad (26)
\end{aligned}
$$

4.4 The non-autonomous HV equation

The composition $w_2 \circ \sigma_{13} \circ \sigma_{12}$ is reduced to

$$
\begin{aligned}
w_2 \circ \sigma_{13} \circ \sigma_{12} \quad & : (x, y; a_1, a_2, a_3, a_4, a_5, a_6, a_7)\\
\mapsto \quad & (-y, x - y - a_5 - \frac{a_1}{y^2} - \frac{a_2}{a_1 y}\ ; \qquad (27)\\
& -a_6, a_7 - 2a_5^2 a_6 - \frac{2a_2 a_6}{a_1}, -a_1, -a_2, a_5, -a_3, -a_4 - 2a_3 a_5^2 + \frac{2a_2 a_3}{a_1})
\end{aligned}
$$

where a_5 can be normalized to be $a_5 = 0$ or 1.

Of course this mapping satisfies the singularity confinement criterion by construction and in the case of $a_2 = a_4 = a_5 = a_7 = 0$ and $a_1 = a_3 = a_6 = a$ it coincides with the HV eq.(1) except their signs. The difference between them comes from the assumption $\overline{a_5} = a_5$. Assuming $\overline{a_5} = -a_5$ under the actions of w_2, σ_{13} and σ_{12}, we have $-w_2$, $-\sigma_{13}$ and $-\sigma_{12}$ as new w_2, σ_{13} and σ_{12} and therefore (27) becomes as follows

$$
\begin{aligned}
w_2 \circ \sigma_{13} \circ \sigma_{12} \quad & : (x, y; a_1, a_2, a_3, a_4, a_5, a_6, a_7)\\
\mapsto \quad & (y, -x + y + a_5 + \frac{a_1}{y^2} + \frac{a_2}{a_1 y}\ ; \qquad (28)\\
& a_6, -a_7 + 2a_5^2 a_6 + \frac{2a_2 a_6}{a_1}, a_1, a_2, -a_5, a_3, a_4 + 2a_3 a_5^2 - \frac{2a_2 a_3}{a_1}).
\end{aligned}
$$

Actually in the case of $a_2 = a_4 = a_5 = a_7 = 0$ and $a_1 = a_3 = a_6 = a$ it coincides with the HV eq.(2).

5 Algebraic entropy

5.1 Algebraic entropy and intersection numbers

In this section we consider the algebraic entropy which has been introduced by Hietarinta and Viallet to describe the complexity of rational mappings [2].

We define the degree of polynomial of one variable $f(t) = \sum_m a_t t^m$ as

$$\deg_t(f(t)) = \max\{m \ ; \ a_m \neq 0\}$$

and the degree of a polynomial of two variable $f(x,y) = \sum_{m,n} a_{m,n} x^m y^n$ as

$$\deg(f(x,y)) = \max\{m+n \ ; \ a_{m,n} \neq 0\}.$$

The degree of an irreducible rational function $P(x,y) = f(x,y)/g(x,y)$, where $f(x,y)$ and $g(x,y)$ are polynomials, is defined by

$$\deg(P) = \max\{\deg f(x,y), \deg g(x,y)\}.$$

The degree of a mapping $\varphi : (x,y) \mapsto (P(x,y), Q(x,y))$, where $P(x,y)$ and $Q(x,y)$ are rational functions, is defined by

$$\deg(\varphi) = \max\{\deg P(x,y), \deg Q(x,y)\}$$

and similarly for $\deg_t(P)$.

The algebraic entropy $h(\varphi)$, where φ is a mapping from $\mathbb{P}^1 \times \mathbb{P}^1$ to itself, is defined by

$$h(\varphi) = \lim_{n\to\infty} \frac{1}{n} \log \deg(\varphi^n)$$

if this limit exists.

Remark. If one would prefer to discuss the mapping in $\mathbb{P}^2$ instead of $\mathbb{P}^1 \times \mathbb{P}^1$, it is sufficient to note that we can relate a mapping φ': $(X,Y,Z) \in \mathbb{P}^2 \mapsto (\overline{X}, \overline{Y}, \overline{Z}) \in \mathbb{P}^2$ with a mapping φ: $(x,y) \in \mathbb{P}^1 \times \mathbb{P}^1 \to (\overline{x}, \overline{y}) \in \mathbb{P}^1 \times \mathbb{P}^1$ by using the relations $x = X/Z, y = Y/Z$ and $\overline{x} = \overline{X}/\overline{Z}, \overline{y} = \overline{Y}/\overline{Z}$ and by reducing to a common denominator. We denote the n-th iterate of φ' by $(f_n(X,Y,Z), g_n(X,Y,Z), h_n(X,Y,Z))$ where f_n, g_n, h_n are polynpmials with the same degree and should be simplified if possible. The algebraic entropy $h(\varphi)$ then coincides with $h(\varphi')$, where $h(\varphi')$ is defined by $\deg(\varphi'^n) = \deg f_n (= \deg g_n = \deg h_n)$ and $\lim_{n\to\infty} \deg(\varphi')$.

We show that we can calculate the degree of the n-th iterate and thus the entropy of mapping by using the theory of intersection numbers.

Let $\{X_i\}$ be a sequence of rational surfaces obtained by blow ups from $\mathbb{P}^1 \times \mathbb{P}^1$ and let $\varphi_i(x,y)$ be an isomorphism from X_{i-1} to X_i. We write the action of

$$\varphi_n \circ \varphi_{n-1} \circ \cdots \circ \varphi_1$$

on $\mathbb{P}^1 \times \mathbb{P}^1$ as $(P_n(x,y), Q_n(x,y))$.

Let us define the curve L in X as $y/x = c$, where $c \in \mathbb{C}$ is a nonzero constant. Notice that

$$\deg_t(P_n(t, ct)) = \deg(P_n(x,y)) \tag{29}$$

holds for generic c.

By the fundamental theorem of algebra, $\deg_t(P_n(t, ct))$ coincides with the intersection number of the curve $x = P_n(t, ct)$ and the curve $x = d$ in $\mathbb{P}^1 \times \mathbb{P}^1$, where $d \in \mathbb{C}$ is a constant.

The class of the curve (or class) $x = d$ is expressed as H_0 in $\mathrm{Pic}(X)$ and hence (see (7)) this intersection number coincides with the coefficient of H_1 of the class of the curve $\varphi^n(L)$ for nonzero constant c. Analogously, the intersection number of the curve $y = Q_n(t, ct)$ and the curve $y = d$ coincides with the coefficient of H_0 of the class of the curve $\varphi^n(L)$ for any nonzero constant c.

Notice that for any isomorphism θ from the rational surface X to the rational surface X' and any divisor D, the relation

$$[\theta(D)] = \theta([D])$$

holds, where $[*]$ means the class of $*$ and in the right hand side θ is the linear operator on $\mathrm{Pic}(X)$ $(= \mathrm{Pic}(X'))$ which is induced by the isomorphism θ.

In our case, we have

$$[\varphi_n \circ \cdots \circ \varphi_1(L)] = \varphi_n \circ \cdots \circ \varphi_1([L])$$

and therefore writing the coefficients of H_0 and H_1 of $\varphi_n \circ \cdots \circ \varphi_1([L])$ as h_n^0, h_n^1, we have the relation

$$\deg_t(P_n(t, ct)) = h_n^1 \quad \deg_t(Q_n(t, ct)) = h_n^0$$

for any nonzero constant $c \in \mathbb{C}$. On the other hand it can be seen that the relation (29) holds for $c = c_0$ such that $[L]$ is invariant under infinitesimal change of c around c_0, i.e. $[L]$ is generic for the parameter c. (Suppose that $P_n(t, ct)$ would suddenly be simplified for certain values of c and that $[L]$ is invariant under infinitesimal change of c. The intersection number of the curve $x = P_n(t, ct)$ and the curve $x = d$ would then change but $\varphi([L])$ itself would still be invariant, which leads to a contradiction. Similarly for the case of $Q_n(t, ct)$.)

Consequently we have the following theorem.

THEOREM **5.1** *Let $\{X_i\}$ be a sequence of rational surfaces obtained by blow ups from $\mathbb{P}^1 \times \mathbb{P}^1$ and let $\varphi_i(x,y)$ be an isomorphism from X_{i-1} to X_i. We denote the action of $\varphi_n \circ \cdots \circ \varphi_1$ on $\mathbb{P}^1 \times \mathbb{P}^1$ by $(P_n(x,y), Q_n(x,y))$. Let $[L]$ be the class of curve $x = cy$ in X_0 such that $[L]$ is generic and let h_n^0, h_n^1 be the coefficients of H_0 and H_1 of $\varphi_n \circ \cdots \circ \varphi_1([L])$. The formula*

$$\deg(P_n(x,y)) = h_n^1 \quad \deg(Q_n(x,y)) = h_n^0.$$

then holds.

Remark. As before if $\{X'_i\}$ is a sequence of rational surfaces obtained by blow ups from $\mathbb{P}^2$ (instead of $\mathbb{P}^1 \times \mathbb{P}^1$) and $\varphi_i(x,y)$ is an isomorphism from X'_{i-1} to X'_i, we can consider the degree of the mapping

$$\varphi'_n \circ \cdots \circ \varphi'_1 : \mathbb{P}^2 \to \mathbb{P}^2.$$

We denote the class of a curve $aX + bY + cZ = 0$ in $\mathbb{P}^2$ by $\mathcal{E}$. Notice that $\mathcal{E}$ is always generic for parameters a, b, c in X_i. Similar to the case of $\mathbb{P}^1 \times \mathbb{P}^1$, we have the fact that the degree of $\varphi'_n \circ \cdots \circ \varphi'_1$ coincides with the coefficient of $\mathcal{E}$ of $\varphi'_n \circ \cdots \circ \varphi'_1(\mathcal{E})$.

5.2 The case of the HV eq.

It is known [2] that the algebraic entropy of the HV eq. φ is equal to $\log(3+\sqrt{5})/2$. Here we shall recover the algebraic entropy of the HV eq. by using the theory of intersection numbers.

The curve $L : x = cy$, where $c \in \mathbb{C}$ is nonzero constant, is expressed by $H_0 + H_1 - E_9$ in $\mathrm{Pic}(X)$ if $c \neq 1$. This fact is easily calculated from the fact that L has intersections only with H_0, H_1 and E_9 at one time.

The action of φ on $\mathrm{Pic}(X)$ is given by (10) or (11). Hence the algebraic entropy of the HV eq., $\lim_{n\to\infty} \frac{1}{n} \log \max\{h_n^0, h_n^1\}$, can be shown to be equal to (by diagonalization of the matrix (11)):

$$\log \max\{|\text{ eigenvalues of (11)}|\} = \log \frac{3+\sqrt{5}}{2}.$$

On the level of the mapping itself, the degrees can be calculated as follows:

$$\begin{aligned}(x,y) &\xrightarrow{\varphi} (y, \frac{-xy^2+y^3+a}{y^2}) \xrightarrow{\varphi} (\deg 3, \deg 9)\\ &\xrightarrow{\varphi} (\deg 9, \deg 25) \xrightarrow{\varphi} (\deg 25, \deg 67) \xrightarrow{\varphi} .\end{aligned} \tag{30}$$

On the other hand, the intersection numbers can be calculated by (10) or (11) as follows:

$$\begin{aligned}H_0 + H_1 - E_9 &\xrightarrow{\varphi} 3H_0 + H_1 - E_5 - E_6 - E_7 - E_8 - E_9\\ &\xrightarrow{\varphi} 9H_0 + 3H_1 + \cdots + (-3)E_9 + \cdots\\ &\xrightarrow{\varphi} 25H_0 + 9H_1 + \cdots + (-7)E_9 + \cdots\\ &\xrightarrow{\varphi} 67H_0 + 25H_1 + \cdots + (-19)E_9 + \cdots\\ &\xrightarrow{\varphi}\end{aligned}$$

which actually coincides with (30).

By the corresponding mapping in $\mathbb{P}^2$, the degrees are

$$\begin{aligned}(X,Y,Z) &\xrightarrow{\varphi'} (Y^3, -XY^2 + Y^3 + aZ^3, Y^2Z) \xrightarrow{\varphi'} \deg 9\\ &\xrightarrow{\varphi'} \deg 27 \xrightarrow{\varphi'} \deg 73 \xrightarrow{\varphi'} .\end{aligned} \tag{31}$$

Using the correspondence $\mathcal{E} = H_0 + H_1 - E_9$ (it is shown that $\mathcal{E} : aX + bY + cZ = 0$ actually has this correspondence in appendix B), we have that the curve has the property

$$\begin{aligned}\varphi^n(\mathcal{E}) \cdot \mathcal{E} &= \varphi^n(H_0 + H_1 - E_9) \cdot (H_0 + H_1 - E_9) \\ &= (h_0 H_0 + h_1 H_1 + e_1 E_1 + \cdots + e_{14} E_{14}) \cdot (H_0 + H_1 - E_9) \\ &= h_0 + h_1 + e_9,\end{aligned}$$

where $e_i \in \mathbb{Z}$ is the coefficient of E_i. Hence we have the sequence of the coefficients of $\mathcal{E}$ as

$$1 \xrightarrow{\varphi'} 3 \xrightarrow{\varphi'} 9 \xrightarrow{\varphi'} 27 \xrightarrow{\varphi'} 73 \xrightarrow{\varphi'} ,$$

which coincides with (31).

Remark. The matrix (11) is an expression of the action φ on $\mathrm{Pic}(X)$ by the basis $\{H_0, H_1, E_1, \cdots, E_{14}\}$. But we already have a better basis for calculation of the degree of φ_n (we may consider linear spaces to be on $\mathbb{C}$ instead of on $\mathbb{Z}$ for this purpose). That is the basis $\{D_0, D_1, \cdots, D_{12}, \alpha_1, \alpha_2, \alpha_3\}$. The reason is that $< D_i >$ and $< \alpha_i >$ are eigenspaces of φ and compliment each other. Moreover the action of φ on $< D_i >$ is just a permutation. Hence it is enough to investigate the action on $< \alpha_i >$ in order to know the level of growth of $\deg(\varphi_n)$.

According to (17), by writing an element of $< \alpha_i >$ as $r_1\alpha_1 + r_2\alpha_2 + r_3\alpha_3$, the action of φ $(= w_2 \circ \sigma_{13} \circ \sigma_{12})$ on $< \alpha_i >$ is expressed as

$$\begin{pmatrix} \overline{r_1} \\ \overline{r_2} \\ \overline{r_2} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 2 & 2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ r_2 \end{pmatrix}. \qquad (32)$$

Remark. In the case of the non-autonomous version (27) the coefficients of H_i and E_i do not change and therefore the degrees and the algebraic entropy do not change, since its action on the Picard group is identical with the action of the original autonomous version.

5.3 The growth of degree of discrete Painlevé equations

It is shown by Sakai [8] that the discrete Painlevé equations can be obtained by the following method.

Let X be a rational surface obtained by blow ups from $\mathbb{P}^2$ such that its anti-canonical divisor $-K_X$ $(= 3\mathcal{E} - E_1 - \cdots - E_9)$ is uniquely decomposed in prime divisors as $-K_X = \sum_{i=1}^{I} m_i D_i$ and satisfies $K_X \cdot D_i = 0$ for all i. This implies that $K_X \cdot K_X = 0$ and therefore X is obtained by 9 points blow ups from $\mathbb{P}^2$ and hence $\mathrm{rankPic(X)} = 10$. One can classify such surfaces according to the type (denoted by R) of Dynkin diagram formed by the D_i (the lattice of R is a sub-lattice of the lattice of $E_8^{(1)}$).

The Cremona isometries of X preserve the sub-lattice $< D_i >$ and its orthogonal sub-lattice with respect to the intersection form. By taking a suitable basis of the orthogonal lattice, $\{\alpha_1, \alpha_2, \cdots, \alpha_J\}$, it is the basis of an extended affine Weyl group and moreover

$\alpha_j \cdot \alpha_j$ does not depend on j. Notice that the intersection number of α_j and K_X is zero, since $\alpha_j \cdot K_X = \alpha_j \cdot \sum m_i D_i = 0$.

The group of Cremona isometries of X is isomorphic to the extended affine Weyl group and each element can be realized as a Cremona transformation on $\mathbb{P}^2$. Each of the discrete Painlevé equations corresponds to a translation of extended affine Weyl group.

The Cartan matrixes of these affine Weyl groups are symmetric and $-K_X$ becomes the canonical central element (and also becomes δ, see § 6.2 § 6.4 in [16]). Hence the action of Painlevé equation on the orthogonal lattice $< \alpha_j >$ is expressed as

$$(\alpha_1, \alpha_2, \cdots, \alpha_J) \mapsto (\alpha_1 + k_1 K_X, \alpha_2 + k_2 K_X, \cdots, \alpha_J + k_J K_X) \tag{33}$$

where $k_j \in \mathbb{Z}$ and $\sum k_j = 0$.

LEMMA **5.2** *Let X, D_i and α_j be as mentioned above. The following formula with respect to the rank:*

$$\mathrm{rank} < \mathrm{D_1, \cdots, D_I}, \alpha_1, \cdots, \alpha_\mathrm{J} >= 9$$

holds.

Proof. Notice that $\{D_1, \cdots, D_I\}$ or $\{\alpha_1, \cdots, \alpha_J\}$ are linearly independent. Suppose $\sum d_i D_i + \sum r_j \alpha_j = 0$, where $d_i, r_j \in \mathbb{C}$. We have $F := -\sum d_i D_i = \sum r_j \alpha_j \in < D_i > \cap < \alpha_j >$. Since F is an element of $< D_i >$, $\alpha_i \cdot (\sum r_j \alpha_j) = 0$ holds for all $1 \le i \le J$. Here the Cartan matrix of the Weyl group is $C := (c_{i,j})_{1 \le i,j \le J}$:

$$c_{i,j} = \frac{2\alpha_i \cdot \alpha_j}{\alpha_i \cdot \alpha_i}$$

and $\alpha_i \cdot \alpha_i$ does not depend on i. Hence it implies

$$C\mathbf{r} = 0, \tag{34}$$

where $\mathbf{r} = (r_1, \cdots, r_J)$. The corank of Cartan matrix of affine type is 1. Hence we obtain $F \in \mathbb{Z}K_X$. It implies the fact that the corank of $< D_1, \cdots, D_I, \alpha_1, \cdots, \alpha_J >$ is 1. □

Let E_9 be an exceptional curve, where "9" means the last blow up.

LEMMA **5.3**

$$\{D_1, \cdots, D_I, \alpha_1, \cdots, \alpha_J, K_X, E_9\}$$

is a basis of Pic(X).

Of course these elements are not independent.

Proof. Suppose $E_9 = \sum d_i D_i + \sum r_j \alpha_j$, where $d_i, r_j \in \mathbb{C}$. Multiplying this equation by K_X, we find $-1 = 0$. The claim of lemma follows from Lemma 5.2. □

Let T be a discrete Painlevé equation. Since T acts on $\{D_i\}$ just as a permutation, there exists l such that T^l acts on $\{D_i\}$ as the identity.

LEMMA 5.4 *There exist integers $z_1, z_2, \ldots, z_J$ such that*

$$T^l(E_9) = E_9 + \sum z_j \alpha_j$$

holds.

Proof. Notice that E_9 has an intersection with only one of the $\{D_i\}$ and without loss of generality we can assume $E_9 \cdot D_1 = 1$. The system of equations $T^l(E_9) \cdot D_1 = 1$, $T^l(E_9) \cdot D_i = 0$ $(i = 2, \ldots I)$ is linear. Hence the solutions of this system are $T^l(E_9) = E_9 + \sum \mathbb{C}\alpha_j$. Of course $T^l(E_9)$ must be an element of $\mathrm{Pic}(X)$ and therefore the coefficients must be integers. $\square$

By Lemma 5.3 and Lemma 5.4 the action of T^l on $\mathrm{Pic}(X)$ is expressed as

$$\begin{aligned} & d_1 D_1 + \cdots + d_I D_I + r_1 \alpha_1 + \cdots + r_J \alpha_J + k K_X + e E_9 \\ \mapsto\ & d_1 D_1 + \cdots + d_I D_I + (r_1 + e z_1)\alpha_1 + \cdots + (r_J + e z_J)\alpha_J \\ & + (k + l r_1 k_1 + \ldots + l r_J k_J) K_X + e E_9 \end{aligned}$$

where $d_i, r_j, k, e \in \mathbb{Z}$. This action is written by the matrix

$$A := \left[\begin{array}{ccc|ccc|c|c} 1 & & & & & & & \\ & \ddots & & & & & & \\ 0 & & 1 & & & & & \\ \hline & & & 1 & & & & z_1 \\ & & & & \ddots & & & \vdots \\ & & & & & 1 & & z_J \\ \hline & & & lk_1 & \cdots & lk_J & 1 & \\ \hline & & & & & & & 1 \end{array}\right], \tag{35}$$

where a blank means 0.

The matrix A^s, where $s \in \mathbb{N}$ is

$$A^s := \left[\begin{array}{ccc|ccc|c|c} 1 & & & & & & & \\ & \ddots & & & & & & \\ 0 & & 1 & & & & & \\ \hline & & & 1 & & & & sz_1 \\ & & & & \ddots & & & \vdots \\ & & & & & 1 & & sz_J \\ \hline & & & slk_1 & \cdots & slk_J & 1 & *_s \\ \hline & & & & & & & 1 \end{array}\right], \tag{36}$$

where $*_s = \frac{1}{2}s(s-1)\sum l k_j z_j$.

Let us start with $\mathcal{E} \in \mathrm{Pic}(X)$ and let $d_1 D_1 + \cdots + d_I D_I + r_1 \alpha_1 + \cdots + r_J \alpha_J + k K_X + e E_9$ be an expression of $\mathcal{E}$. We obtain the following theorem.

THEOREM 5.5 *For all discrete Painlevé equations the order of degree of the n-th iterate is at most $O(n^2)$.*

Proof. The degree of the Painlevé equation T as a birational mapping of $\mathbb{P}^2$ coincides with the coefficient of $\mathcal{E}$ in $T^n(\mathcal{E})$ as an action on $\mathrm{Pic}(X)$. Because the coefficients of

$$\begin{aligned} & T^{sl}(\mathcal{E}) \\ = \ & \sum_i d_i D_i + \sum_j (r_j + s z_j e)\alpha_j + + \left(sl \sum_j k_j r_j + k + \frac{1}{2}s(s-1)le \sum_j k_j z_j \right) K_X + E_9, \end{aligned} \tag{37}$$

where $n = sl$, increase at most with the order $O(s^2)$, the coefficient of $\mathcal{E}$ also increases at most $O(n^2)$. □

6 Some other examples

We present some examples of rational mappings which satisfy the singularity confinement criterion and some of which have positive algebraic entropy.

6.1 Example 1

Let w_1, w_2, w_3 and a_i be as in § 4. First we consider the mapping $w_1 \circ w_2$

$$\begin{aligned} & w_1 \circ w_2 : (x, y; a_1, a_2, a_3, a_4, a_5, a_6, a_7) \\ \mapsto & \Big(x - \frac{a_1}{y^2} - \frac{a_2}{a_1 y}, y - \frac{a_3}{(x - a_1/y^2 - a_2/(a_1 y))^2} - \frac{a_4 - 2a_2a_3/a_1}{a_3(x - a_1/y^2 - a_2/(a_1 y))}, \ ; \\ & -a_1, a_2 + \frac{2(a_1a_4 - 2a_2a_3)}{a_3}, -a_3, a_4 - \frac{2a_2a_3}{a_1}, a_5, a_6, a_7 + \frac{2a_2a_6}{a_1} + 2(a_4 - 2\frac{a_2a_3}{a_1})\frac{a_6}{a_3} \Big) \end{aligned}$$

This mapping has the following properties.

1)This mapping satisfies the singularity confinement criterion.
2)The order of the n-th iterate of mapping is $O(n^2)$ (easily seen from the action on $< \alpha >$).
3)The actions on the parameters a_5, a_6, a_7 can be ignored.

This mapping is nothing but one of the discrete Painlevé equations, since the surface obtained by blowing down the curves $E_9, E_{10}, \cdots, E_{14}$ in X is also the space of initial values and the type of Dynkin diagram corresponding to the irreducible components of anti-canonical divisor is $D_7^{(1)}$ with the symmetry $A_1^{(1)}$. Actually the irreducible components of anti-canonical divisor are

$$\begin{aligned} & E_1 - E_2,\ E_2 - E_3,\ E_3 - E_4,\ E_5 - E_6,\ E_6 - E_7,\ E_7 - E_8, \\ & H_0 - E_1 - E_2,\ H_1 - E_5 - E_6 \end{aligned}$$

and the root basis of orthogonal lattice is

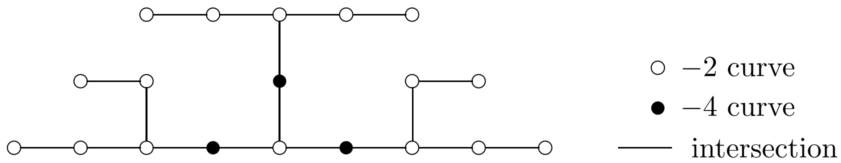
$$\begin{aligned} \alpha_1 &= 2H_1 - E_1 - E_2 - E_3 - E_4, \\ \alpha_2 &= 2H_0 - E_5 - E_6 - E_7 - E_8. \end{aligned}$$

Next we consider the mapping $w_3 \circ w_2 \circ w_1$. This mapping is almost identical to the nonautonomous HV eq. after 3 steps. Actually the latter becomes $w_2 \circ w_3 \circ w_1$.

At last we consider the mapping $w_2 \circ w_3 \circ w_2 \circ w_1$. This mapping satisfies the singularity confinement criterion and its algebraic entropy is $17 + 12\sqrt{2}$.

6.2 Example 2

We consider the following diagram as irreducible components of the anti-canonical divisor.



This diagram is realized by the sequence of blow ups from $\mathbb{P}^1 \times \mathbb{P}^1$ as follows

$$
\begin{aligned}
(x,y) \quad & \xleftarrow[\mu_1]{(\infty,0)} \left(\frac{1}{xy}, y\right) \xleftarrow[\mu_2]{(0,0)} \left(\frac{1}{xy^2}, y\right) \xleftarrow[\mu_3]{(0,0)} \left(\frac{1}{xy^2}, xy^3\right) \\
& \xleftarrow[\mu_4]{(0,a_1)} \left(\frac{1}{xy^2}, xy^2(xy^3-a_1)\right) \xleftarrow[\mu_5]{(0,a_2)} \left(\frac{1}{xy^2}, xy^2(xy^2(xy^3-a_1)-a_2)\right) \\
& \xleftarrow[\mu_6]{(0,a_3)} \left(\frac{1}{xy^2}, xy^2(xy^2(xy^2(xy^3-a_1)-a_2)-a_3)\right),
\end{aligned}
$$

$$
\begin{aligned}
(x,y) \quad & \xleftarrow[\mu_7]{(0,\infty)} \left(x, \frac{1}{xy}\right) \xleftarrow[\mu_8]{(0,0)} \left(x, \frac{1}{x^2y}\right) \xleftarrow[\mu_9]{(0,0)} \left(x^3y, \frac{1}{x^2y}\right) \\
& \xleftarrow[\mu_{10}]{(a_4,0)} \left(x^2y(x^3y-a_4), \frac{1}{x^2y}\right) \xleftarrow[\mu_{11}]{(a_5,0)} \left(x^2y(x^2y(x^3y-a_4)-a_5), \frac{1}{x^2y}\right) \\
& \xleftarrow[\mu_{12}]{(a_6,0)} \left(x^2y(x^2y(x^2y(x^3y-a_4)-a_5), \frac{1}{x^2y}\right)
\end{aligned}
$$

and

$$
(x,y) \xleftarrow[\mu_{13}]{(\infty,\infty)} \left(\frac{1}{x}, \frac{x}{y}\right) \xleftarrow[\mu_{14}]{(0,1)} \left(\frac{1}{x}, x(\frac{x}{y}-1)\right) \xleftarrow[\mu_{15}]{(0,a_7)} \left(\frac{1}{xz}, z\right),
$$

where we denote $z := x(x/y-1)-a_7$,

$$
\begin{aligned}
& \xleftarrow[\mu_{16}]{(0,0)} \left(\frac{1}{xz^2}, z\right) \xleftarrow[\mu_{17}]{(0,0)} \left(\frac{1}{xz^2}, xz^3\right) \xleftarrow[\mu_{18}]{(0,a_8)} \left(\frac{1}{xz^2}, xz^2(xz^3-a_8)\right) \\
& \xleftarrow[\mu_{19}]{(0,a_9)} \left(\frac{1}{xz^2}, xz^2(xz^2(xz^2-a_8)-a_9)\right) \xleftarrow[\mu_{20}]{(0,a_{10})} \left(\frac{1}{xz^2}, xz^2(xz^2(xz^2(xz^3-a_8)-a_9)-a_{10})\right).
\end{aligned}
$$

Similar to the case of the HV eq.(§ 4), we obtain

$$
\begin{aligned}
w_2 \quad : \quad & (x, y : a_1, a_2, \cdots, a_{10}) \\
\mapsto \quad & \Bigg(x, y - \frac{a_4}{x^3} - \frac{a_5}{a_4x^2} - (\frac{a_6}{a_4^2} - \frac{a_5^2}{a_4^3})\frac{1}{x} \ : \\
& a_1, a_2, a_3 + \frac{3a_1^2a_5^2}{a_4^3} - \frac{3a_1^2a_6}{a_4^2}, -a_4, a_5, -a_6, a_7, a_8, a_9, a_{10} - \frac{3a_5^2a_8^2}{a_4^3} + \frac{3a_6a_8^2}{a_4^2}\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{13} \quad : \quad & (x, y; a_1, a_2, \cdots, a_{10}) \\
\mapsto \quad & \left(x, x-y-a_7 \ ; \ a_8, a_9, a_{10} - 3a_7^2a_8^2, -a_4, a_5, -a_6, a_7, a_1, a_2, a_3 + 3a_1^2a_7^2\right),
\end{aligned}
$$

$$\begin{aligned}\sigma_{12} \quad &: \quad (x,y;a_1,a_2,\cdots,a_{10}) \\ &\mapsto \quad (-y,-x\ ;\ a_4,-a_5,a_6,a_1,-a_2,a_3,a_7,-a_8,a_9,-a_{10}+6a_7^2a_8^2)\end{aligned}$$

and finally

$$\begin{aligned}&w_2\circ\sigma_{13}\circ\sigma_{12}:(x,y;a_1,a_2,\cdots,a_{10}) \ \mapsto \\ &\Big(-y,-y+x-a_7-\frac{a_1}{y^3}-\frac{a_2}{a_1y^2}+(\frac{-a_3}{a_1^2}+\frac{a_2^2}{a_1^3})\frac{1}{y}\ ; \quad -a_8,a_9, \\ &-a_{10}+3a_8^2a_7^2-\frac{3a_2^2a_8^2}{a_1^3}+\frac{3a_8^2a_3}{a_1^2},a_1,-a_2,a_3,a_7,a_4,-a_5,a_6+3a_4^2a_7^2+\frac{3a_2^2a_4^2}{a_1^3}-\frac{3a_3a_4^2}{a_1^2}\Big)\end{aligned}$$

In the case $a_i=0$ for $i=2,3,5,6,9,10$, this mapping reduces to

$$w_2\circ\sigma_{13}\circ\sigma_{12}:(x,y;a_1,a_4,a_7,a_8) \quad \mapsto \quad (-y,x-y-a_7+\frac{a_1}{y^3};-a_8,a_1,a_7,a_4). \tag{38}$$

We present some basic properties of this mapping.

The Picard group of the space of initial values is

$$\mathrm{Pic}(X)=\mathbb{Z}H_0+\mathbb{Z}H_1+\mathbb{Z}E_1+\cdots+\mathbb{Z}E_{20}$$

and the canonical divisor of X is

$$K_X=-2H_0-2H_1+E_1+\cdots+E_{20}.$$

The irreducible components of the anti-canonical divisor are

$$\begin{gathered}E_1-E_2,\ E_2-E_3,\ E_3-E_4,\ E_4-E_5,\ E_5-E_6, \\ E_6-E_7,\ E_7-E_8,\ E_8-E_9,\ E_9-E_{10},\ E_{11}-E_{12}, \\ E_{12}-E_{13},\ E_{15}-E_{16},\ E_{16}-E_{17},\cdots,E_{19}-E_{20}, \\ H_0-E_1-E_2-E_3-E_{13},\ H_1-E_5-E_6-E_7-E_{13},\ E_{14}-E_{15}-E_{16}-E_{17}\end{gathered}$$

and the root basis is

$$\begin{aligned}\alpha_1 &= 3H_1-E_1-E_2-E_3-E_4-E_5-E_6, \\ \alpha_2 &= 3H_0-E_7-E_8-E_9-E_{10}-E_{11}-E_{12}, \\ \alpha_3 &= 3H_0+3H_1-3E_{13}-3E_{14}-E_{15}-E_{16}-E_{17}-E_{18}-E_{19}-E_{20}.\end{aligned}$$

The Cartan matrix $2(\alpha_i\cdot\alpha_j)/(\alpha_i\cdot\alpha_i)$ is

$$\begin{bmatrix} 2 & -3 & -3 \\ -3 & 2 & -3 \\ -3 & -3 & 2 \end{bmatrix}. \tag{39}$$

This Cartan matrix is not finite, affine nor hyperbolic type.
The algebraic entropy is $2+\sqrt{3}$.

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Appendix

A Uniqueness of the decomposition of the anti-canonical divisor

THEOREM **A.1** *Let X be the space of initial values of the HV eq. obtained in Section 2. The anti-canonical class of divisors $-K_X$ can be reduced uniquely to prime divisors as*

$$D_0 + 2D_1 + D_2 + D_3 + 2D_4 + D_5 + 3D_6 + D_7 + 2D_8 + D_9 + 2D_{10} + 2D_{11} + 2D_{12} \quad (40)$$

Proof. Suppose that $-K_X$ is reduced to prime divisors as $-K_X = \sum_{i=1}^{m} f_i F_i$, where F_i is a prime divisor and f_i is a positive integer. F_i has the form

$$h_0^{(i)} H_0 + h_1^{(i)} H_1 - \sum_{j=1}^{14} e_j^{(i)} E_j,$$

where h_j and e_j are nonnegative integers and $h_j \leq 2$ and moreover h_0 or h_1 is strictly positive (remember that $-K_X = 2H_0 + 2H_1 - \sum_{j=1}^{14} E_j$), or in the case the curve is included in the total transform of a blow up point:

$$E_1 - E_2, E_2 - E_3, E_3 - E_4, E_4, E_5 - E_6, E_6 - E_7, E_7 - E_8, E_8,$$
$$E_9 - E_{10}, E_{10} - E_{11} - E_{12}, E_{11} - E_{12}, E_{12} - E_{13}, E_{13} - E_{14}, E_{14}.$$

Let us first suppose that there does not exist F_i such that $F_i = E_{13} - E_{14}$. Then there exists a F_i such that F_i has the form as $h_0 H_0 + h_1 H_1 - \sum_{j=1}^{14} e_j E_j$ where e_{14} is strictly positive. This means that F_i has an intersection point with E_{14} and therefore this divisor passes the point of the 14-th blow up before the blow up. Namely,

$$\left(\frac{1}{x^2(x/y-1)}, x^2(\frac{x}{y}-1)(x^3(\frac{x}{y}-1)^2 - a)\right) = (0,0)$$

as $x, y \to \infty$. Denoting $u = 1/x, v = 1/v$, we have $u^3/(v-u) = 0$ and $(v-u)(v^2 - 2v + 1 - au^5)/u^8 = 0$ as $u, v \to 0$. This implies $v^2 - 2v + 1 - au^5 = o(\frac{u^8}{v-u})$ and hence we have $h_0 \geq 5$, which is a contradiction.

Similarly there does not exist a divisor which has the form $h_0 H_0 + h_1 H_1 - \sum_{j=1}^{13} e_j E_j$ where e_{12} or e_{13} is strictly positive and $h_j \leq 2$.

Now, we can consider the case where there does not exist a F_i such that $F_i = E_{14}$. Then according to the above result, there should exist integers j, k such that $F_j = E_{13} - E_{14}, F_k = E_{12} - E_{13}$ and $f_j = 1, f_k = 2$ (in the absence of a prime divisor E_{14}, the form of $-K_X$ forces $f_j = 1$ and subsequently $f_k = 2$ in order to have the correct E_{13} dependence). Considering the new coefficient of E_{12}, we have that the sum of the coefficients of $E_{11} - E_{12}$ and $E_{10} - E_{11} - E_{12}$ has to be 3.

Then we have the following possibilities:
i) the coefficient of $E_{11} - E_{12}$ is 0;
ii) the coefficient of $E_{11} - E_{12}$ is 1;
iii) the coefficient of $E_{11} - E_{12}$ is 2;
iv) the coefficient of $E_{11} - E_{12}$ is 3.

In the case i),ii),iii) or iv) the coefficient of E_{11} is $-3, -1, 1$ or 3 respectively. Hence i) is impossible. To pass the point of the 11-th blow up, a divisor whose class has the form $h_0 H_0 + h_1 H_1 - \sum_{j=1}^{11} e_j E_j$ must satisfy the equation $u = 0$ and $(v-u)/u^2 = 0$. This implies $h_0 \geq 2, h_1 \geq 1$ and $e_{11} = 1$ and therefore iii) and iv) are impossible. Consequently the coefficient of $E_{11} - E_{12}$ has to be 1 and the coefficient of $E_{10} - E_{11} - E_{12}$ has to be 2.

Along the same lines, considering the coefficient of E_{10}, we have the following possibilities:
a) the coefficient of $E_9 - E_{10}$ is 0, in which case the coefficient of E_9, E_{10} is $0, 2$;
b) the coefficient of $E_9 - E_{10}$ is 1, in which case the coefficient of E_9, E_{10} is $1, 1$;
c) the coefficient of $E_9 - E_{10}$ is 2, in which case the coefficient of E_9, E_{10} is $2, 0$;
d) the coefficient of $E_9 - E_{10}$ is 3, in which case the coefficient of E_9, E_{10} is $3, -1$.

To pass the point of the 10-th blow up, a divisor whose class has the form $h_0 H_0 + h_1 H_1 - \sum_{j=1}^{11} e_j E_j$ must satisfy the equation $u = 0$ and $v/u = 1$. It implies two possibilities: i) $h_0 \geq 1, h_1 \geq 1$, $e_9 = e_{10} = 1$ and $e_2, e_3, e_4, e_6, e_7, e_8 = 0$ (since if e_2 would be strictly positive, the intersection of the divisor $h_0 H_0 + h_1 H_1 - \sum_{j=1}^{11} e_j E_j$ and $x = \infty$ $(= H_0)$ is greater than or equal to 3, which is impossible) or ii) $h_0 H_0 + h_1 H_1 - \sum_{j=1}^{11} e_j E_j = 2H_0 + 2H_1 - 2E_9 - 2E_{10}$. Therefore a) is impossible. In the case b) the coefficient of E_9 becomes -1 (or $1, 3$ in the cases c), d) respectively) and the coefficients of H_0 and H_1 become equal to 2 (or become greater than or equal to 1, become 0 in the cases c), d) respectively).

In the case b) the coefficients of $E_1, E_2, \cdots, E_8$ is 0 and hence the remaining contribution $-E_1 - \cdots - E_8$ must be a sum of the terms:

$$E_1 - E_2, E_2 - E_3, E_3 - E_4, E_4, E_5 - E_6, E_6 - E_7, E_7 - E_8, E_8 \tag{41}$$

with nonnegative integer coefficients. Obviously this is impossible. In the case c) the remaining contribution has the form $b'_0 H_0 + b'_1 H_1 - b_1 E_1 - \cdots - b_8 E_8 - 2E_9$, where $b'_0, b'_1 = 0$ or 1, $b_2, b_3, b_4, b_6, b_7, b_8 = 1$ and $b_1, b_2 = 0$ or 1 and hence it must be a sum of $H_0 - E_1 - E_2 - E_9$, $H_1 - E_5 - E_6 - E_9$ and (41) with nonnegative integer coefficients. By straight forward discussion it can be seen that this case is also impossible.

The coefficient of $E_9 - E_{10}$ therefore has to be equal to 3 and hence the coefficient of E_9 has to be 3 as well, from which point on it can be easily seen that the remaining contributions to $-K_X$ are uniquely determined.

Similarly, starting from the supposition that there does exist a F_i such that $F_i = E_{14}$, similar proof can be given. $\square$

B The space of initial values constructed by blowing up $\mathbb{P}^2$

We consider the construction of the space of initial values for the HV eq. $\varphi : \mathbb{P}^2 \to \mathbb{P}^2$

$$\varphi : (X, Y, Z) \mapsto (Y^3, -XY^2 + Y^3 + aZ^3, Y^2 Z).$$

This mapping is reduced from (2) by the change of variables $x = X/Z, y = Y/Z$.

The space of initial values X' becomes isomorphic to X where X is the space of initial values in the case of $\mathbb{P}^1 \times \mathbb{P}^1$. Denoting a class of the curve $aX + bY + cZ = 0$ by $\mathcal{E}$, where $a, b, c \in \mathbb{C}$ (a bears no relation with a in φ), we have a correspondence between the bases of their Picard groups as follows

$$\mathrm{Pic}(X') = \mathbf{Z}\mathcal{E} + \mathbf{Z}E'_p + \mathbf{Z}E'_q + \mathbf{Z}E'_1 + \cdots + \mathbf{Z}E'_8 + \mathbf{Z}E'_{10} + \mathbf{Z}E'_{11} + \mathbf{Z}E'_{14}$$

$$\begin{gathered}\mathcal{E} = H_0 + H_1 - E_9, E'_p = H_0 - E_9, E'_q = H_1 - E_9,\\ E'_i = E_i \quad \text{for } i = 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14.\end{gathered}$$

The intersection form is

$$\mathcal{E} \cdot \mathcal{E} = 1, \quad E'_i \cdot E'_j = -\delta_{i,j}, \quad \mathcal{E} \cdot E'_i = 0$$

for all $i, j \in \{p, q, 1, 2, \cdots, 14\}$.
The irreducible components of the anti-canonical divisor are

$$\begin{gathered}E'_1 - E'_2,\ E'_2 - E'_3,\ E'_3 - E'_4,\ E'_5 - E'_6,\ E'_6 - E'_7,\ E'_7 - E'_8,\\ \mathcal{E} - E'_p - E'_q - E'_{10},\ E'_{11} - E'_{12},\ E'_{12} - E'_{13},\ E'_{13} - E'_{14}\\ E'_p - E'_1 - E'_2,\ E'_q - E'_5 - E'_6,\ E'_{10} - E'_{11} - E'_{12}\end{gathered}$$

and the root basis is

$$\begin{aligned}\alpha_1 &= 2\mathcal{E} - 2E'_p - E'_1 - E'_2 - E'_3 - E'_4,\\ \alpha_2 &= 2\mathcal{E} - 2E'_q - E'_5 - E'_6 - E'_7 - E'_8,\\ \alpha_3 &= 2\mathcal{E} - 2E'_{10} - E'_{11} - E'_{12} - E'_{13} - E'_{14}.\end{aligned}$$

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