Discrete dynamical systems associated with the configuration space of 8 points in $\mathbb{P}^3(\mathbb{C})$

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Abstract

A 3 dimensional analogue of Sakai's theory concerning the relation between rational surfaces and discrete Painlevé equations is studied. For a family of rational varieties obtained by blow-ups at 8 points in general position in $\mathbb{P}^3$, we define its symmetry group using the inner product that is associated with the intersection numbers and show that the group is isomorphic to the Weyl group of type $E_7^{(1)}$. By parametrizing the configuration space by means of elliptic curves, the action of the Weyl group and the dynamical system associated with a translation are explicitly described. As a result, it is found that the action of the Weyl group on $\mathbb{P}^3$ preserves a one parameter family of quadratic surfaces and that it can therefore be reduced to the action on $\mathbb{P}^1 \times \mathbb{P}^1$.

1 Introduction

Relations between Painlevé equations and rational surfaces were first studied by Okamoto[12]. He showed that for each Painlevé equation, by elimination of the singularity of the equation, the solutions can be regularly extended into a family of rational surfaces. Such a family of rational surfaces is called *the space of initial conditions* for the Painlevé equation. Conversely, it is clarified by Saito and Takano et al. [16, 14] that for a given space of initial conditions the Hamilton system of a Painlevé equation can be determined.

Since the singularity confinement method was introduced by Grammaticos et al. [6], the discrete Painlevé equations have been studied extensively ([13] for example). Emphasizing the fact that each discrete Painlevé equation preserves a family of rational surfaces, Sakai constructed the discrete Painlevé equations from families of rational surfaces (called generalized Halphen surfaces) and subsequently classified them. Such a family of rational surfaces is called *the space of initial conditions* for that discrete Painlevé equation. A generalized Halphen surface can be seen to be isomorphic to a surface obtained by 9 blow-ups from $\mathbb{P}^2$. Sakai's classification also shows that the spaces of initial conditions for the discrete Painlevé equations include those for continuous ones. Furthermore, the largest symmetry arises when the 9 points are in general position; all other symmetries are in the case where the points are in some special position. Each Painlevé equation is obtained as a translation associated with the corresponding affine Weyl group (extended by the automorphisms of the associated Dynkin diagram). Moreover, if the space of initial conditions is that of a continuous one, its Weyl group coincides with the group of that equation's Bäcklund transformations.

The aforementioned results all concern non-autonomous dynamical systems. There also exist some studies that deal with autonomous ones. For example, in the continuous

case Adler and van Moerbeke have studied Painlevé manifolds [1] and in the discrete case various authors have studied the relations between dynamical systems and the automorphism groups of manifolds ([4, 5] for example).

Sakai's procedure for describing discrete Painlevé equations is closely related to the studies on the Cremona isometry carried out by Coble et al. [2, 3] and these two approaches coincide in the case of 9 points in general position. Whereas in the case of points in general position the Weyl group is generated by the standard Cremona transformation and exchanges of the points, in the degenerate case its generators can be constructed by changing the blow-down structures. Concerning this point one has to cite the pioneering research by Looijenga [10].

Dolgachev and Ortland also studied the case of 3 (or higher) -dimensional rational varieties [3] : here (for example) the affine Weyl group of type $E_7^{(1)}$ appears in the case where 8 generic points in $\mathbb{P}^3$ are blown-up. If the number of points is less than 8 the Weyl group is finite and it is indefinite if the number of points is larger than 8. However, in the 3-dimensional case the action of each element of the Weyl group cannot be lifted to an isomorphism between rational varieties, obtained by blow-ups at some points. Dolgachev and Ortland call such a map a pseudo-isomorphism.

In this paper we study the symmetry, parametrization of the configuration space and the associated discrete dynamical systems for the family of rational varieties obtained by blow-ups at 8 ordered points in general position in $\mathbb{P}^3$.

In section 2, we reconstruct the argument of Dolgachev and Ortland. We consider birational automorphisms of the family of varieties such that (i) each of them acts as an automorphism on the configuration space (ii) for rational varieties on the configuration space it preserves the "inner product" of the Picard group $\mathrm{Pic}(X)$. Here, the inner product is defined by using the intersection numbers and the canonical divisor K_X as $(D, D') := D \cdot D' \cdot (-\frac{1}{2}K_X)$ for $D, D' \in \mathrm{Pic}(X)$. It is shown that the resulting symmetry group is the Weyl group of type $E_7^{(1)}$. This group coincides with that of Dolgachev and Ortland.

In section 3, parametrization of the configuration space is discussed. Although there is a straightforward parametrization, it is difficult to describe the action of the Weyl group by using this and to see the properties of the resulting dynamical systems. In this paper we therefore use a parametrization in terms of elliptic curves. Quadratic surfaces passing through the 8 points we consider form at least one parameter family. Here the 8 points are on the intersection curve of the pencil of surfaces. Normalizing the pencil, one obtains a parametrization of the configuration space.

In section 4, we describe the action of the Weyl group obtained in section 2 in normalized coordinates. In order to calculate the concrete action we apply a 3-dimensional analogue of the period map which is introduced by Looijenga [10] and Sakai [15] for surfaces.

In section 5, we construct a birational dynamical system in $\mathbb{P}^3$ by using the action obtained in section 4. Such systems are obtained corresponding to translations associated to the Weyl group. We describe one of them explicitly.

Is section 6, it is shown that the action of the Weyl group preserves each member of the pencil of quadratic surfaces and that it can therefore be reduced to an action on $\mathbb{P}^1 \times \mathbb{P}^1$. The reduced action of the Weyl group of type $E_7^{(1)}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ coincides with the action of a sub-group of the Weyl group of type $E_8^{(1)}$, which is the symmetry of the family of (the most) general Halphen surfaces.

Section 7 is devoted to conclusions and discussions.

2 Symmetry

Let $X(4,8)$ denote the configuration space of ordered 8 points in $\mathbb{P}^3(\mathbb{C})$ such that every 4 points are not on the same plane:

$$PGL(4,\mathbb{C}) \backslash \left\{ \left(\begin{array}{cccc} x_1 & x_1 & \cdots & x_8 \\ y_1 & y_2 & \cdots & y_8 \\ z_1 & z_3 & \cdots & z_8 \\ w_1 & w_4 & \cdots & w_8 \end{array} \right) \in (\mathbb{C}^4)^8 \left| \begin{array}{l} \text{every } 4 \times 4 \\ \text{minor determinant} \\ \text{is nonzero} \end{array} \right. \right\} \Big/ (\mathbb{C}^\times)^8, \quad (1)$$

where two configurations are identified if one can be transformed to the other by a projective transformation. We also denote the 3-dimensional rational variety obtained by successive blowing-up at distinct 8 points $P_i(x_i : y_i : z_i : w_i)$ by $X_{P_1,\cdots,P_8}$ (or simply by X) and the family of all $X_{P_1,\cdots,P_8}$'s, where $\{P_1,\cdots,P_8\} \in X(4,8)$, by $\{X_{P_1,\cdots,P_8}\}$. $X(4,8)$ is called the parameter space.

Let $\mathrm{Pic}(X)$ be the Picard group of the variety $X = X_{P_1,\cdots,P_8}$ (the additive group of isomorphism classes of invertible sheaves $\simeq$ the additive group of linear equivalence classes of divisors). We have

$$\begin{aligned} \mathrm{Pic}(X) &= \mathbb{Z}E \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}E_3 \oplus \mathbb{Z}E_4 \oplus \mathbb{Z}E_5 \oplus \mathbb{Z}E_6 \oplus \mathbb{Z}E_7 \oplus \mathbb{Z}E_8, \quad (2) \\ K_X &= -4E + 2(E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7 + E_8), \end{aligned}$$

where E denotes the total transform of the divisor class of the plane in $\mathbb{P}^3$ and E_i denotes the total transform of the exceptional divisor generated by the blow-up at P_i. We often identify the lattices $\mathrm{Pic}(X_{P_1,\cdots,P_8})$'s by (2). For the following argument we define δ as

$$\delta := -\frac{1}{2}K_X \quad (3)$$

and the inner product of $\mathrm{Pic}(X)$ as

$$(D, D') := D \cdot D' \cdot (-\frac{1}{2}K_X)$$

for $D, D' \in \mathrm{Pic}(X)$, where for $D, D', D'' \in \mathrm{Pic}(X)$, $D \cdot D' \cdot D''$ denotes the intersection number $\mathrm{Pic}(X) \times \mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z}$.

We consider the group (written as $\mathrm{Gr}(\{X\})$) of birational transformations on the family $\{X_{P_1,\cdots,P_8}\}$ such that
i) $\varphi : X(4,8) \to X(4,8)$ is an automorphism; here, we denote $\varphi(\{P_1,\cdots,P_8\})$ by $\{P'_1,\cdots,P'_8\}$;
ii) for any $\{P_1,\cdots,P_8\} \in X(4,8)$ the map $\varphi : X_{P_1,\cdots,P_8} \dashrightarrow X_{P'_1,\cdots,P'_8}$ is a birational map preserving the inner product of $\mathrm{Pic}(X)$, i.e. $(\varphi_*(D), \varphi_*(D')) = (D, D')$ for any $D, D' \in \mathrm{Pic}(X_{P_1,\cdots,P_8})$, where $\varphi_* : \mathrm{Pic}(X_{P_1,\cdots,P_8}) \to \mathrm{Pic}(X_{P'_1,\cdots,P'_8})$ is the push-forard action of φ.

Theorem 2.1. $\mathrm{Gr}(\{X\})$ *is the affine Weyl group of type* $E_7^{(1)}$.

Remark 2.1. Dolgachev and Ortland [3] have shown that the affine Weyl group of type $E_7^{(1)}$ acts on $\{X_{P_1,\cdots,P_8}\}$ and satisfies i) and ii)': for any $\{P_1,\cdots,P_8\} \in X(4,8)$ the map $\varphi : X_{P_1,\cdots,P_8} \dashrightarrow X_{P'_1,\cdots,P'_8}$ is a pseudo-isomorphism, i.e. isomorphism in co-dimension one. Moreover, those maps act on $\mathrm{Pic}(X)$ and its Poincaré dual as the root lattice Q and the coroot lattice $\check{Q}$ respectively. Hence we have an isomorphism $\nu : Q \to \check{Q}$ and the corresponding biliniear form on Q. However, to author's knowledge, it has not been proved that there is no maps satisfying i) and ii)' other than the affine Weyl group of type $E_7^{(1)}$.

Moreover, without concrete birational maps, it is difficult to find root and coroot bases and the isomorphism ν. Although we either do not prove the uniqueness in the category of pseudo-isomorphisms, our method makes it possible to find the symmetries and the bilinear forms on $\mathrm{Pic}(X) \supset Q$ from some families of rational varieties themselves.

Before the proof we describe some formulae for the intersection numbers. For elements in $\mathrm{Pic}(X)$ the intersection numbers are given by

$$\begin{aligned} &E \cdot E \cdot E = 1, \quad E \cdot E \cdot E_i = 0, \\ &E \cdot E_i \cdot E_i = 0, \quad E_i \cdot E_i \cdot E_i = 1, \\ &E_i \cdot E_j \cdot D = 0 \quad (i \neq j, \forall D \in \mathrm{Pic}(X)) \ . \end{aligned}$$

Hence the inner product is given by

$$(E, E) = 2, \quad (E, E_i) = 0, \quad (E_i, E_j) = -\delta_{i,j} \ . \tag{4}$$

Lemma 2.1. *If a birational map $\varphi : X_{P_1,\cdots,P_8} \dashrightarrow X_{P'_1,\cdots,P'_8}$ preserves the inner product, then $\varphi_* : \mathrm{Pic}(X_{P_1,\cdots,P_8}) \to \mathrm{Pic}(X_{P'_1,\cdots,P'_8})$ is an isomorphism of lattices.*

Proof. Note that φ_* $(= (\varphi^{-1})^*)$ can be considered to be a linear transformation to itself. Assume that there exists a nonzero divisor $D \in \mathrm{Pic}(X_{P_1,\cdots,P_8})$ such that $\varphi_*(D) = 0$. Since D is nonzero, there exists a divisor $D' \in \mathrm{Pic}(X_{P_1,\cdots,P_8})$ such that $(D, D') \neq 0$. On the other hand, $\varphi_*(D) = 0$ and hence $(\varphi_*(D), \varphi_*(D')) = 0$, which contradicts the assumption that φ preserves the inner product. □

By this lemma, each $\varphi_* : \mathrm{Pic}(X_{P_1,\cdots,P_8}) \to \mathrm{Pic}(X_{P'_1,\cdots,P'_8})$ is an isomorphism and preserves
a) the inner product;
b) the anti-canonical divisor $-K_X$, i.e. $\varphi_*(-K_{X_{P_1,\cdots,P_8}}) = -K_{X_{P'_1,\cdots,P'_8}}$, since φ is a pseudo-isomorphism;
c) the effectiveness of divisors.

We call an automorphism of $\mathrm{Pic}(X)$ which preserves a), b), c) a (3-dimensional) Cremona isometry.

Proof of Theorem 2.1

As in the 2-dimensional case, Theorem 2.1 is proved by investigating the group of Cremona isometries and realizing the corresponding birational transformations.

Let $\varphi \in \mathrm{Gr}(\{X\})$. Since φ_* preserves K_X and the inner product, it also preserves the orthogonal complement of K_X

$$Q(\alpha) := < \alpha_0, \alpha_1, \cdots, \alpha_7 >_{\mathbb{Q}},$$

where

$$\begin{aligned} &\alpha_0 = E_1 - E_2, \ \alpha_1 = E_2 - E_3, \ \cdots, \ \alpha_6 = E_7 - E_8, \\ &\alpha_7 = E - E_1 - E_2 - E_3 - E_4. \end{aligned} \tag{5}$$

Claim 2.1. *The basis $< \alpha_0, \alpha_1, \cdots, \alpha_7 >$ of the linear space $K_X^{\perp}$ generates the lattice $< \alpha_0, \alpha_1, \cdots, \alpha_7 >_{\mathbb{Z}}$, i.e.*

$$< \alpha_0, \alpha_1, \cdots, \alpha_7 >_{\mathbb{Q}} \cap \mathrm{Pic}(X) = < \alpha_0, \alpha_1, \cdots, \alpha_7 >_{\mathbb{Z}}$$

holds.

Proof. We show that the left hand side includes the right hand side. Let $a_0\alpha_0+\cdots+a_7\alpha_7 \in \mathrm{Pic}(X)$. Since the coefficient of E_8 is an integer, a_6 is also an integer. Since the coefficient of E_7 is an integer, $-a_5+a_6$ is also an integer, and so is a_5. Along the same line all a_i's are integers. □

From this claim, φ_* is an automorphism of the sub-lattice $Q(\alpha)$ and preserves the inner product. The matrix defined by using the inner product as

$$(c_{i,j})_{i,j} \quad := \quad 2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)} \tag{6}$$

is the affine Cartan matrix of type $E_7^{(1)}$. We denote the affine Weyl group generated by

$$r_j(\alpha) := r_{\alpha_j}(\alpha) = \alpha - 2\frac{(\alpha_j,\alpha)}{(\alpha_j,\alpha_j)}\alpha_j \quad (\alpha \in Q(\alpha)) \tag{7}$$

$(i=0,1,\cdots,7)$ by $W(E_7^{(1)})$. By the following proposition by Kac we have the fact that the

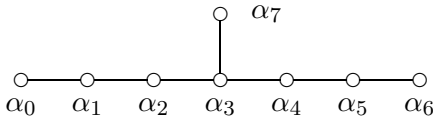

Figure 1: the Dynkin diagram of type $E_7^{(1)}$

group of isometries of $Q(\alpha)$ each of which preserves the inner product is $\pm\mathrm{Aut}(\mathrm{Dynkin}) \ltimes W(E_7^{(1)})$ (which is written as $\pm\widetilde{W}$).

Proposition 2.1. ([8] §5.10) *If the generalized Cartan Matrix c_{ij} is a symmetric matrix of finite, affine, or hyperbolic type, then the group of all automorphisms of $Q(\alpha)$ preserving the bilinear form is $\pm\widetilde{W}$.*

Note that

$$\delta \quad = \quad -\frac{1}{2}K_X = \alpha_0+2\alpha_1+3\alpha_2+4\alpha_3+3\alpha_4+2\alpha_5+\alpha_6+2\alpha_7 \tag{8}$$

is preserved by $W(E_7^{(1)})$ and the automorphism of the Dynkin diagram. In our case, since φ_* preserves the anti-canonical divisor $-K_X$, each element of $-\mathrm{Aut}(\mathrm{Dynkin}) \ltimes W(E_7^{(1)})$ is not a Cremona isometry.

Claim 2.2. *The automorphism of the Dynkin diagram is not a Cremona isometry.*

Proof. There exists only one automorphism of the Dynkin diagram, which is the involution exchanging $\alpha_0,\alpha_1,\alpha_2$ and $\alpha_6,\alpha_5,\alpha_4$ respectively.

Denoting the actions of these involutions on $\mathrm{Pic}(X)$ by $D \mapsto \overline{D}$, we have

$$\begin{aligned}
\overline{E} &= E+3E_1-E_2-E_3-E_4-E_5-E_6-E_7-E_8+4\overline{E_8}\\
\overline{E_1} &= E_1-E_8+\overline{E_8}\\
\overline{E_2} &= E_1-E_7+\overline{E_8}\\
&\vdots\\
\overline{E_7} &= E_1-E_2+\overline{E_8}.
\end{aligned}$$

Set $\overline{E_8} = eE + e_1E_1 + \cdots + e_8E_8$. From

$$\begin{aligned}
(\overline{E}, \overline{E_8}) &= (E, E_8) = 2e - 3e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 + e_8 - 4 = 0 \\
(\overline{E_1}, \overline{E_8}) &= (E_1, E_8) = -e_1 + e_8 - 1 = 0 \\
&\vdots \\
(\overline{E_7}, \overline{E_8}) &= (E_7, E_8) = -e_1 + e_2 - 1 = 0,
\end{aligned}$$

we have $e = -2e_1 - 3/2$, which contradicts the assumption that e is an integer. □

Now we have the fact that the actions of Cremona isometries on $Q(\alpha)$ are included by $W(E_7^{(1)})$.

Claim 2.3. *The action of* $W(E_7^{(1)})$ *are uniquely extended onto* $\mathrm{Pic}(X)$ *as*

$$r_j(D) = D - 2\frac{(\alpha_j, D)}{(\alpha_j, \alpha_j)}\alpha_j \quad (D \in \mathrm{Pic}(X)). \tag{9}$$

Proof. Let s and s' be Cremona isometries such that the action of s is identical to that of s' on $Q(\alpha)$. We show $s' \circ s^{-1} =$ Identity on $\mathrm{Pic}(X)$. Since $\{E_1, \alpha_0, \alpha_1, \cdots, \alpha_7\}$ is a basis of $\mathrm{Pic}(X)$, we can set

$$s' \circ s^{-1}(E_1) = e_1E_1 + a_0\alpha_0 + a_1\alpha_1 + \cdots + a_7\alpha_7.$$

From $(s' \circ s^{-1}(E_1), \delta) = (E_1, \delta) = 1$ and $(\alpha_i, \delta) = 0$ (for $\forall i$), we have $e_1 = 1$. Since

$$(s' \circ s^{-1}(E_1), \alpha_i) = (E_1, \alpha_i) \Longleftrightarrow (s' \circ s^{-1}(E_1) - E_1, \alpha_i) = 0$$

holds,

$$-(c_{i,j})_{0 \le i,j \le 7}\mathbf{a} = 0$$

holds, where $(c_{i,j})_{0 \le i,j \le 7}$ is the Cartan matrix of type $E_7^{(1)}$ (6). Hence we have $s' \circ s^{-1}(E_1) = E_1 + z\delta$ ($z \in \mathbb{Z}$). Finally, from $(s' \circ s^{-1}(E_1), s' \circ s^{-1}(E_1)) = -1 + 2z = -1$, we have $z = 0$. Hence it has been shown that $s' \circ s^{-1}$ does not change the basis of $\mathrm{Pic}(X)$ and therefore $s' \circ s^{-1} =$ Identity. □

By this claim, the actions of simple reflections on $\mathrm{Pic}(X)$ are given by

$$\begin{aligned}
r_i : \quad & E_{i+1} \mapsto E_{i+2} \\
& E_{i+2} \mapsto E_{i+1} \quad (\text{for } 0 \le i \le 6) \\
r_7 : \quad & E \mapsto 3E - 2E_1 - 2E_2 - 2E_3 - 2E_4 \\
& E_k \mapsto E - E_1 - E_2 - E_3 - E_4 + E_k \quad (\text{for } 1 \le k \le 4),
\end{aligned}$$

where preserved elements are omitted. Finally, we have the claim of the theorem by the following claim.

Claim 2.4. *The action of each element of* $W(E_7^{(1)})$ *on* $\mathrm{Pic}(X)$ *is uniquely realized as an element of* $\mathrm{Gr}(\{X\})$.

Proof. It is enough to show for the simple reflections r_i's.
i) in the case of $0 \le i \le 6$. Since $E \mapsto E$, the action on $\mathbb{P}^3$ is linear. This is nothing but

the projective transformation $PGL(4)$. Hence, without loss of generality, we can assume that the action is the identity. Since r_i exchanges E_{i+1} and E_{i+2}, we have

$$\begin{aligned} r_i : \quad & ((P_1, \cdots, P_{i+1}, P_{i+2}, \cdots, P_8); \mathbf{x}) \in X(4.8) \times \mathbb{P}^3 \\ & \mapsto ((P_1, \cdots, P_{i+2}, P_{i+1}, \cdots, P_8); \mathbf{x}). \end{aligned} \tag{10}$$

Here, we have described the action $r_i : X_{P_1,\ldots,P_8} \to X_{P'_1,\ldots,P'_8}$ in terms of the coordinate $\mathbf{x}$ of $\mathbb{P}^3$.

ii) in the case of r_7. Using $PGL(4)$, we may assume

$$P_1 = P'_1 = (1:0:0:0), \cdots, P_4 = P'_4 = (0:0:0:1) \tag{11}$$

without loss of generality. Since $E \mapsto 3E - 2E_1 - 2E_2 - 2E_3 - 2E_4$, the action on $\mathbb{P}^3$: $(x:y:z:w) \mapsto (x':y':z':w')$ is in the 3rd degree. Moreover, from $E_1 \leftrightarrow E - E_2 - E_3 - E_4$, we have $(1:0:0:0) \leftrightarrow x' = 0$. Since similar facts hold for the case of E_2, E_3 and E_4, considering the degree with respect to x, y, z, w, we have $(x':y':z':w') = (ayzw : bzwx : cwxy : dxyz)$, where $a, b, c, d \in \mathbb{C}^\times$. Furthermore, we can normalize it as $a = b = c = d = 1$ preserving (11). This is nothing but the standard Cremona transformation of $\mathbb{P}^3$. Hence the action of r_7 on $\{X\}$ is given by the composition of maps:

$$\begin{aligned} & ((P_1, P_2, P_3, P_4), (P_5, P_6, P_7, P_8); \mathbf{x}) \in X(4.8) \times \mathbb{P}^3 \\ \mapsto \quad & \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, P_{1234}^{-1}(P_5, P_6, P_7, P_8); P_{1234}^{-1} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \right) \\ & := ((P'_1, \cdots, P'_8); \mathbf{x}') \\ \mapsto \quad & \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} x'_{2,5}x'_{3,5}x'_{4,5} & \cdots & x'_{2,8}x'_{3,8}x'_{4,8} \\ x'_{3,5}x'_{4,5}x'_{1,5} & \cdots & x'_{3,8}x'_{4,8}x'_{1,8} \\ x'_{4,5}x'_{1,5}x'_{2,5} & \cdots & x'_{4,8}x'_{1,8}x'_{2,8} \\ x'_{1,5}x'_{2,5}x'_{3,5} & \cdots & x'_{1,8}x'_{2,8}x'_{3,8} \end{pmatrix}; \begin{pmatrix} y'z'x' \\ z'w'x' \\ w'x'y' \\ x'y'z' \end{pmatrix} \right), \end{aligned} \tag{12}$$

where P_{1234} denotes the square matrix (P_1, P_2, P_3, P_4). □

□

3 Parametrization of the configuration space

In this section we discuss parametrization of the configuration space $X(4,8)$ by $PGL(4)$, i.e. how to choose representative elements. Notice that without a good parametrization it is difficult to see the concrete action of the group and properties of associated dynamical systems. For example, although $X(4,8)$ is easily parametrized as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & y_6 & y_7 & y_8 \\ 0 & 0 & 1 & 0 & 1 & z_6 & z_7 & z_8 \\ 0 & 0 & 0 & 1 & 1 & w_6 & w_7 & w_8 \end{pmatrix}, \tag{13}$$

the action of the Weyl group on this coordinate becomes complicated. We parametrize it in terms of elliptic curves.

Note the following lemma.

Lemma 3.1. *Let $P_1, P_2, \cdots, P_8$ be 8 points in $\mathbb{P}^3$. Quadratic surfaces passing through $P_1, P_2, \cdots, P_8$ form at least one parameter family.*

Proof. It is clear from the fact that quadratic surfaces passing through $P_1, P_2, \cdots, P_8$ and arbitrary point P_9 in $\mathbb{P}^3$ is given by the equation

$$\begin{vmatrix} x^2 & y^2 & z^2 & w^2 & xy & yz & zw & wx & xz & yw \\ x_1^2 & y_1^2 & z_1^2 & w_1^2 & x_1y_1 & y_1z_1 & z_1w_1 & w_1x_1 & x_1z_1 & y_1w_1 \\ & & & & \vdots & & & & & \\ x_9^2 & y_9^2 & z_9^2 & w_9^2 & x_9y_9 & y_9z_9 & z_9w_9 & w_9x_9 & x_9z_9 & y_9w_9 \end{vmatrix} = 0,$$

where $P_i = (x_i : y_i : z_i : w_i)$ (if the left hand side is identically zero, exchange one of the points for a generic point). □

The pencil of quadratic surfaces passing through $P_1, P_2, \cdots, P_8$ can be written as

$$\mathbf{x}^t(\alpha A + \beta B)\mathbf{x} = 0 \qquad (\mathbf{x} \in \mathbb{P}^3) \tag{14}$$

by 4×4 complex symmetric matrices A, B and $(\alpha : \beta) \in \mathbb{P}^1$. Normalizing (14) by $PGL(4)$ (cf. [18]), we have the following theorem, which provides a parametrization of $X(4, 8)$. The proof of this theorem will be given in the last part of this section.

Theorem 3.1. *Each element of $X(4, 8)$ can be parametrized so that $P_1, P_2, \cdots, P_8$ are on the intersection curve(s) of one of the following 3 type pencils of quadratic surfaces.*
(i)

$$\begin{array}{ll} (E) & x^2 - zw = 0 \\ (F) & y^2 - 4xw + g_2xz + g_3z^2 = 0 \end{array}$$

(ii)

$$\begin{aligned} xy - zw &= 0 \\ xw - z^2 &= 0 \end{aligned}$$

(iii)

$$\begin{aligned} xy - zw &= 0 \\ 4x^2 - 2zw + w^2 &= 0\ . \end{aligned}$$

Moreover,

(i-1) *in the case where the intersection curve is non-singular ($\Delta = 27g_3^2 - g_2^3 \neq 0$), P_i can be parameterized as*

$$P_i = (\wp(u_i) : \wp'(u_i) : 1 : \wp^2(u_i)), \qquad u_i \in \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau), \tag{15}$$

where $\wp(u)$ is Wierstrass $\wp$ function with the periods $(1, \tau)$.
(i-2) *in the case where $\Delta = 0$ and $g_2 \neq 0$, the intersection curve can be renormalized to the intersection curve of*

$$\begin{aligned} x^2 - zw &= 0 \\ y^2 - 4w(x + hz) &= 0 \end{aligned}$$

and P_i can be parameterized as

$$P_i = \left(\frac{4hu_i}{(1-u_i)^2} : \frac{-8h^{3/2}u_i(1+u_i)}{(1-u_i)^3} : 1 : \frac{(4hu_i)^2}{(1-u_i)^4} \right), \qquad u_i \in \mathbb{P}^1 \setminus \{0, \infty\} \ . \quad (16)$$

(i-3) *in the case where $g_2 = g_3 = 0$, the intersection curve can be renormalized to the intersection curve of*

$$\begin{aligned} x^2 - zw &= 0 \\ y^2 - 4xw &= 0 \end{aligned}$$

and P_i can be parameterized as

$$P_i = \left(u_i^{-2} : -2u_i^{-3} : 1 : u_i^{-4}\right), \qquad u_i \in \mathbb{P}^1 \setminus \{\infty\} \ . \quad (17)$$

(ii) *the intersection consists of 2 curves $\{(0 : s : 0 : t) \mid s : t \in \mathbb{P}^1\}$ and $\{(s^3 : t^3 : s^2t : st^2) \mid s : t \in \mathbb{P}^1\}$.*
(iii) *the intersection consists of 2 curves $\{(0 : s : 0 : t) \mid s : t \in \mathbb{P}^1\}$ and $\{(2st^2 : s(s^2+4t^2) : t(s^2+4t^2) : 2s^2t) \mid s : t \in \mathbb{P}^1\}$.*

Remark 3.1. The parameterizations of (i-2) and (i-3) of Theorem 3.1 are chosen so that the period map becomes simple.

From the proof in case (i-1) of this theorem we have the following corollary.

Corollary 3.1. *If the intersection curve of surfaces passing through $P_1, P_2, \cdots, P_8$ is non singular, $X(4,8)$ can be parametrized as $P_1, P_2, \cdots, P_8$ are on the intersection curve of 2 surfaces*

$$\begin{aligned} \theta_{00}^2 x^2 - \theta_{01}^2 y^2 - \theta_{10}^2 z^2 &= 0 \\ \theta_{10}^2 y^2 - \theta_{01}^2 z^2 - \theta_{00}^2 w^2 &= 0, \end{aligned} \quad (18)$$

where we write $\theta_{ij} := \theta_{ij}(0)$ for the theta function $\theta_{ij}(u)$ with the fundamental periods $(1, \tau)$ and therefore P_i can be parameterized as

$$P_i = (\theta_{00}(2u_i) : \theta_{01}(2u_i) : \theta_{10}(2u_i) : \theta_{11}(2u_i)). \quad (19)$$

Corollary 3.2. *$X(4,8)$ restricted to case* (i-1),(i-2) *or* (i-3) *of Theorem 3.1 is isomorphic to*
(i-1)

$$\{(u_1, \cdots, u_8, \tau) \in (\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau))^8 \times (\mathbb{H}/SL(2,\mathbb{Z})) \mid u_i + u_j + u_k + u_l \not\equiv 0\}, \quad (20)$$

where $1 \leq i, j, k, l \leq 8$ are different each other and $\mathbb{H}$ is the upper half of the complex plane;
(i-2)

$$\{(u_1, \cdots, u_8) \in (\mathbb{P}^1 \setminus \{0, \infty\})^8 \mid u_i u_j u_k u_l \neq 1\}; \quad (21)$$

(i-3)

$$\{(u_1, \cdots, u_8) \in (\mathbb{P}^1 \setminus \{\infty\})^8 \mid u_i + u_j + u_k + u_l \neq 0\}. \quad (22)$$

Proof. It is enough to show that P_i, P_j, P_k, P_l are on the same plane if and only if $u_i + u_j + u_k + u_l = 0$ holds ($u_iu_ju_ku_l = 1$ holds in case (i-2)). Notice thatP_i, P_j, P_k, P_l are on the same plane if and only if

$$\begin{vmatrix} x_i & y_i & z_i & w_i \\ x_j & y_j & z_j & w_j \\ x_k & y_k & z_k & w_k \\ x_l & y_l & z_l & w_l \end{vmatrix} = 0 \tag{23}$$

holds.
· In case (i-1). When (23) is considered to be a rational function of u_i, the origin is the unique pole of order 4. By Abel's Theorem the sum of zero points are 0. Here, $u_i = u_j, u_k, u_l$ are zero points and hence the other zero point is $u_i = -u_j - u_k - u_l$.
· In case (i-2) or (i-3). When (23) is considered to be a rational function of u_i, $u_i = 1$ (the origin in case (i-3)) is the unique pole of order 4 and therefore there are 4 zero points. It is easily shown that $u_i = u_j, u_k, u_l, (u_ju_ku_l)^{-1}$, ($u_i = u_j, u_k, u_l, -u_j - u_k - u_l$ in case (i-3)) are those 4 points by substitution. □

A similar argument leads the following theorem concerning the dimension of linear system $|-\frac{1}{2}K_X|$ of rational variety $X = X_{P_1,\cdots,P_8}$.

Theorem 3.2. *In case (i) of theorem 3.1,* $\dim(|-\frac{1}{2}K_X|)$ *is 2 if and only if, in case* (i-1) *or* (i-2)

$$u_1 + u_2 + \cdots + u_8 = 0 \tag{24}$$

holds and in case (i-1)

$$u_1u_2\cdots u_8 = 1 \tag{25}$$

holds.

Since each surface in $|-\frac{1}{2}K_X|$ is a rational projective elliptic surface, $\dim|-\frac{1}{2}K_X| \leq 2$ holds and if the equivalent conditions of Theorem 3.2 is satisfied, X is an elliptic variety.

Proof. Case (i-1).
($\Rightarrow$). If $\dim(|-\frac{1}{2}K_X|) \geq 2$, there exists a surface $D \in |-\frac{1}{2}K_X|$ which does not include the intersection curve of (i-1). Let P_9 and P_{10} be generic points on D. D is described as

$$\begin{vmatrix} x^2 & y^2 & z^2 & w^2 & xy & yz & zw & wx & xz & yw \\ x_1^2 & y_1^2 & z_1^2 & w_1^2 & x_1y_1 & y_1z_1 & z_1w_1 & w_1x_1 & x_1z_1 & y_1w_1 \\ & & & & \vdots & & & & & \\ x_7^2 & y_7^2 & z_7^2 & w_7^2 & x_7y_7 & y_7z_7 & z_7w_7 & w_7x_7 & x_7z_7 & y_7w_7 \\ x_9^2 & y_9^2 & z_9^2 & w_9^2 & x_9y_9 & y_9z_9 & z_9w_9 & w_9x_9 & x_9z_9 & y_9w_9 \\ x_{10}^2 & y_{10}^2 & z_{10}^2 & w_{10}^2 & x_{10}y_{10} & y_{10}z_{10} & z_{10}w_{10} & w_{10}x_{10} & x_{10}z_{10} & y_{10}w_{10} \end{vmatrix} = 0. \tag{26}$$

By Bézout's theorem the intersection of D and the curve of (i-1) is the 8 points $P_1, P_2, \cdots, P_8$, which are given by the zero points of (26) with $(x : y : z : w) = (\wp(u) : \wp(u)' : 1 : \wp(u)^2)$. The left hand side of the equation is an elliptic functions of u and has the unique pole $u = 0$ of order 8. Since the 8 points $u = u_1, u_2, \cdots, u_8$ are zero points, by Abel's theorem we have $u_1 + u_2 + \cdots + u_8 = 0$.
($\Leftarrow$) Assume $u_1 + u_2 + \cdots + u_8 = 0$. By considering the zero points of the same elliptic

function, it is shown that the intersection of a generic quadratic surface D passing through 7 points $P_1, P_2, \cdots, P_7$ and the curve of (i-1) is the 8 points $P_1, P_2, \cdots, P_8$. Hence D is an element of $|-\frac{1}{2}K_X|$. The dimension to choose such D is 2 or higher.
In case (i-2) or (i-3), it is enough to change the argument about Abel's theorem as the proof of Corollary 3.2. □

Proof of Theorem 3.1

Since the later part is easy, we show only the former part.

We consider normalization of (14) by $PGL(4)$. Note that P^{-1} in $PGL(4)$ acts on the pencil as

$$\mathbf{x}^t(\alpha A + \beta B)\mathbf{x} = 0 \rightarrow \mathbf{x}^t(\alpha P^t A P + \beta P^t B P)\mathbf{x} = 0.$$

Note also the following facts.

Fact 1. Any complex symmetric matrix can be diagonalized to the form $\mathrm{diag}(1, \cdots, 1, 0, \cdots, 0)$ by PGL.
Fact 2. The $n \times n$ identity matrix is not changed by orthonormal matrices.
Fact 3. Since "two complex symmetric matrices are similar if and only if they are similar via a complex orthonormal similarity," if two matrices have the same Jordan normal form, they are mapped each other by some complex orthonormal matrix. (pp.212 in [7]).

Assume that there exists $(s:t)$ such that $\mathrm{rank}(sA + tB) = 2$, by Fact 1 the matrix is normalized to $\mathrm{diag}(1, 1, 0, 0)$ and the defining equation can be factorized. Hence, 4 or more points in the 8 points on $\mathbf{x}^t(\alpha A + \beta B)\mathbf{x} = 0$ are on the same plane, which contradicts the assumption of the configuration space. So we have $\mathrm{rank}(sA + tB) \geq 3$ for all $(s:t) \in \mathbb{P}^1$.

Lemma 3.2. *If* $\mathrm{rank}(sA + tB) \geq 3$ *for all* $(s:t) \in \mathbb{P}^1$*, there exists* $(s:t) \in \mathbb{P}^1$ *such that* $\mathrm{rank}(sE + tF) = 4$*.*

Proof. Without loss of generality we can assume $A = \mathrm{diag}(1, 1, 1, 0)$. We normalize B by 3×3 matrix. Since the sizes of Jordan blocks of the 3×3 submatrix of B should be $(1, 1, 1), (2, 1)$ or (3), by Fact 3, we can set B as

$$\begin{pmatrix} a & 0 & 0 & e \\ 0 & b & 0 & f \\ 0 & 0 & c & g \\ e & f & g & d \end{pmatrix}, \quad \begin{pmatrix} a+1 & \sqrt{-1} & 0 & e \\ \sqrt{-1} & a-1 & 0 & f \\ 0 & 0 & b & g \\ e & f & g & d \end{pmatrix}, \quad \begin{pmatrix} a & 1 & \sqrt{-1} & e \\ 1 & a & 0 & f \\ \sqrt{-1} & 0 & a & g \\ e & f & g & d \end{pmatrix}.$$

Assume $\mathrm{rank}(sA + tB) = 3$ for all $(s, t) \in \mathbb{P}^1$. In the first case, we have $d = e = f = g = 0$ and therefore the rank becomes 2 or less for some t, which is a contradiction. Along the same line, in the second or the third case it can be shown that the rank becomes 2 or less, which is a contradiction. Hence there exists $(s:t) \in \mathbb{P}^1$ such that $\mathrm{rank}(sA+tB) = 4$. □

From the above lemma, we may assume $A =$ Identity and $\mathrm{rank}B = 3$. The Jordan normal form of B has the following 5 possibilities:

$$(\mathrm{i}-1) \qquad \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$
\begin{aligned}
(\mathrm{i}-2)\quad & \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a+1 & \sqrt{-1} & 0 \\ 0 & \sqrt{-1} & a-1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
(\mathrm{i}-3)\quad & \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & a & 1 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
(\mathrm{ii})\quad & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 1 & \sqrt{-1} \\ 0 & 0 & \sqrt{-1} & -1 \\ 1 & \sqrt{-1} & -1 & \sqrt{-1} \\ \sqrt{-1} & -1 & \sqrt{-1} & 1 \end{pmatrix}, \\
(\mathrm{iii})\quad & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & a & 1 \\ 0 & 0 & 0 & a \end{pmatrix} \sim \begin{pmatrix} 1 & \sqrt{-1} & 0 & 0 \\ \sqrt{-1} & -1 & 0 & 0 \\ 0 & 0 & 2 & \sqrt{-1} \\ 0 & 0 & \sqrt{-1} & 0 \end{pmatrix},
\end{aligned}
$$

where the right hand side matrices are similar to B except the proportional constants. We replace B by the right hand side matrices.

· Case (i-1). Since $\mathrm{rank}(sA+tB) \geq 3$, a, b, c are not zero and different each other. By replacing the basis of pencil $\mathbf{x}^t(\alpha A + \beta B)\mathbf{x} = 0$, we may assume $A = \mathrm{diag}(0, a, b, c)$ and $B = \mathrm{diag}(d, e, f, 0)$. Moreover, by the actions of diagonal matrices, we can set $A = \mathrm{diag}(0, 1, 1, 1)$ and $B = \mathrm{diag}(a, a, b, 0)$. Finally, by multiplying a constant to B, we can set $A = \mathrm{diag}(0, 1, 1, 1), B = \mathrm{diag}(1, 1, a, 0)$ $(a \neq 0, 1)$. On the other hand if $\Delta = g_2^3 - 27g_3^2$ is not zero, using $e_1 = \wp(w_1/2), e_2 = \wp(w_2/2), e_3 = \wp((w_1+w_2)/2)$, (F) can be written as

$$y^2 - 4xw + 4(e_1+e_2+e_3)x^2 - 4(e_1e_2+e_1e_3+e_2e_3)xz + 4e_1e_2e_3z^2 = 0.$$

The matrices

$$
\begin{aligned}
E &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}, \\
F &= \begin{pmatrix} 4(e_1+e_2+e_3) & 0 & -2(e_1e_2+e_1e_3+e_2e_3) & -2 \\ 0 & 1 & 0 & 0 \\ -2(e_1e_2+e_1e_3+e_2e_3) & 0 & 4e_1e_2e_3 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

are normalized to $E' = \mathrm{diag}(0, 1, 1, 1), F' = \mathrm{diag}(1, 1, \lambda, 0)$ via

$$
P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{-1} & -1 \\ 0 & 0 & \sqrt{-1} & 1 \end{pmatrix},
$$

where

$$
\begin{aligned}
\lambda &= \frac{e_2 - e_3}{e_1 - e_3} \\
&= \frac{3^{1/3}(3\sqrt{-1}+\sqrt{3})g_2 - (-3\sqrt{-1}+\sqrt{3})(9g_3 + \sqrt{-3g_2^3+81g_3^2})^{2/3}}{-3^{1/3}(-3\sqrt{-1}+\sqrt{3})g_2 + (3\sqrt{-1}+\sqrt{3})(9g_3 + \sqrt{-3g_2^3+81g_3^2})^{2/3}} \qquad (27)
\end{aligned}
$$

is the λ function. By suitably replacing P, λ can be changed to $1/\lambda, 1-\lambda, 1/(1-\lambda), \lambda/(\lambda-1), (\lambda-1)/\lambda$ (by using the fact that E' and F' are simultaneously decomposed to the eigenspaces, it can also be shown that λ cannot be other than these.) Note that λ is invariant under the action $(w_1, w_2) \mapsto (sw_1, sw_2) \Leftrightarrow (g_2, g_3) \mapsto (g_2/s^4, g_3/s^6)$. We show that there exist corresponding g_2 and g_3 when $\lambda \neq 0, 1, \infty$ ($\Delta = 0$ if $\lambda = 0, 1$ or ∞). Setting

$$y = \frac{\lambda - 1/2 - \sqrt{3}\sqrt{-1}/2}{\lambda - 1},$$

we have

$$y = \frac{3\sqrt{-1}(1/2 - \sqrt{3}/2)g_2}{g_2 - (3\sqrt{3}g_3 + \sqrt{-g_2^3 + 27g_3^2})^{(2/3)}}.$$

We show that there exist corresponding g_2 and g_3 when $y \neq 1/2 + \sqrt{3}\sqrt{-1}/2, 1, \infty$. Setting $g_2 = a, (3\sqrt{3}g_3 + \sqrt{-g_2^3 + 27g3^2})^{(2/3)} = 3b^2$, we have

$$y = \frac{3\sqrt{-1}(1/2 - \sqrt{3}/2)a}{a - 3b^2}.$$

Hence, there exist corresponding $a, b \in \mathbb{C}$ for arbitrary $y \in \mathbb{C}$. Since $g_2 = a, g_3 = a^3/(54b^3) + b^3/2$, we can find the corresponding g_2 and g_3 when $b \neq 0$. If $b = 0$, we have $g_2 = g_3 = 0$, which is not in the case considered here.

· Case (i-2). Set

$$P_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-\sqrt{-1}}{\sqrt{a}} & \frac{-1}{(a-1)\sqrt{a}} & 0 \\ 0 & 0 & \frac{\sqrt{-1}\sqrt{a}}{2(1-a)} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{-a}{a-1}} \end{pmatrix},$$

$$\begin{aligned} A' &= P_2^t(P_1AP_1 - P_1BP_1)P_2, \\ B' &= \frac{1-a}{a}P_2^tP_1BP_1P_2, \end{aligned}$$

Then $A' = \mathrm{diag}(1, 1, 1, 0)$ and the Jordan normal form of B' is

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{28}$$

On the other hand, if $\Delta = 0$, setting

$$P_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{-1} & -1 \\ 0 & 0 & \sqrt{-1} & 1 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} \sqrt{2\sqrt{3g_2}} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$E' = P_2P_1^tAP_1P_2,$$

$$F' = -\frac{2}{3}P_2(P_1^tBP_1\frac{\sqrt{3}}{4\sqrt{g_2}} - P_1^tAP_1)P_2,$$

we have $E' = \mathrm{diag}(1,1,1,0)$ and (28) as the Jordan normal form of F'. Hence it has been shown that the two pencils are equivalent modulo $PGL(4)$.
· Case (i-3). If $g_2 = g_3 = 0$, setting

$$P = \begin{pmatrix} 0 & 1 & \frac{2\sqrt{-1}}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{-1}}{\sqrt{5}} & \frac{-9}{\sqrt{5}} \\ 0 & 0 & \frac{\sqrt{-1}}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix},$$
$$E' = P^t(A+B)P, \qquad F' = P^tAP,$$

we have $E' = $ Identity and

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

as the Jordan normal form of F'. Hence it has been shown that the two pencils are equivalent modulo $PGL(4)$.
· Case (ii). Setting

$$P = \begin{pmatrix} 1 & 1/2 & 0 & 0 \\ -\sqrt{-1} & \sqrt{-1}/2 & 0 & 0 \\ 0 & 0 & -1 & 1/2 \\ 0 & 0 & -\sqrt{-1} & -\sqrt{-1}/2 \end{pmatrix},$$
$$A' = P^tAP,$$
$$B' = P^tBP,$$

we have

$$A' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad B' = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}.$$

· Case (iii). Setting P, A' and B' as in case (ii), we have

$$A' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad B' = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

□

4 Period map and the action of the Weyl group

In this section we describe the action of $W(E_7^{(1)})$ in case (i) of Theorem 3.1 in the normalized coordinate system. For this purpose, it is enough to normalize the simple reflections obtained in Claim 2.4, but it is not easy calculus. Thus, first, we define a linear map χ_X

from the lattice $Q(\alpha)$ to $\mathbb{C}$ (which has ambiguity corresponding to the periods as discussed later). Next, we compute the action by using the fact that χ is invariant under the action of $W(E_7^{(1)})$, i.e. $\chi_X(\alpha) = \chi_{w(X)}(w(\alpha))$ for $\alpha \in Q(\alpha)$ and $w \in W(E_7^{(1)})$. This method is an analogue of that of the period map essentially introduced by Looijenga for surfaces.

Period map and the action on the intersection curve

In the following, we shall discuss only in case (i-1) of Theorem 3.1. For case (i-2) and (i-3), we shall write the results only. Replace $x/z, y/z, w/z$ by x, y, w. Let $D_1, D_2 \in -\frac{1}{2}K_X$ the divisors determined by the proper tranforms of $(E), (F)$ in Theorem 3.1. We denote the set of piece-wise smooth singular 3-chains in $X - D_1 - D_2$ by $S(X - D_1 - D_2)$. Using the holomorphic 3-form on $X \setminus (D_1 \cup D_2)$

$$\omega = \frac{c\ dx \wedge dy \wedge dw}{(x^2 - w)(y^2 - 4xw + g_2x + g_3)} \tag{29}$$

(the constant $c \in \mathbb{C}^\times$ is determined later), we define the map $\chi_X : S(X - D_1 - D_2) \to \mathbb{C}$ by the paring $\int_\Gamma \omega$, $(\Gamma \in S(X - D_1 - D_2))$.

Let C denote the elliptic curve $D_1 \cap D_2$. We define a map $Q(\alpha) \to S(X - D_1 - D_2)/H_1(C, \mathbb{Z})$. For this purpose, it is enough to define the map for the basis α_i's.
· In the case of $0 \leq i \leq 6$. We have $\alpha_i = E_{i+1} - E_{i+2}$. Let C_i^{re} be a real curve on C from $C \cap E_{i+1}$ (which is expressed as $u = u_{i+1}$ in the coordinate u) to $C \cap E_{i+2}$ $(u = u_{i+2})$. Here, C has ambiguity of $H_1(C, \mathbb{Z}) \simeq \mathbb{Z} + \mathbb{Z}\tau$. Let $\varepsilon > 0$ be a sufficiently small number and $\Gamma_i \in S(X - D_1 - D_2)$ the set of points such that $|x^2 - w| = \varepsilon$, $|y^2 - 4xw + g_2x + g_3| = \varepsilon$ and x is in the projection of C_i^{re} to the x coordinate.
· In the case of $i = 7$. We have $\alpha_7 = (E - E_1 - E_2 - E_3) - E_4$. Let C_7^{re} be a real curve on C from $C \cap (E - E_1 - E_2 - E_3)$ ($u = -u_1 - u_2 - u_3$, cf. Lemma 3.2) to $C \cap E_4$ $(u = u_4)$. Let $\Gamma_7 \in S(X - D_1 - D_2)$ be the set of points such that $|x^2 - w| = \varepsilon$, $|y^2 - 4xw + g_2x + g_3| = \varepsilon$ and x is in the projection of C_7^{re} to the x coordinate.

Remark 4.1. As in 2-dimensional case, we can take Γ_i from $H_3(X - D_1 - D_2, \mathbb{Z})$. Let F_1, F_2 be divisors such that α_i is written as $\alpha_i = F_1 - F_2$ as above. For our purpose, it is enough to add singular 3-chains in F_1 and F_2 to the above Γ_i. Here, the extended part is included by 2-dimensional algebraic subvariety and hence the effect for the integration is zero.

By the composition $Q(\alpha) \to S(X - D_1 - D_2)/H_1(C, \mathbb{Z}) \to \mathbb{C}$, the map $\chi_X : Q(\alpha) \to \mathbb{C}$ is determined modulo the image of $H_1(C, \mathbb{Z})$. Let π_x denote the projection to the x coordinate. By residue theorem, we have

$$\begin{aligned}
\chi_X(\alpha_i) &= \int_{\Gamma_i} \omega \\
&= c \int_{\substack{|x^2 - w| = \varepsilon \\ |y^2 - 4xw + g_2x + g_3| = \varepsilon \\ x \in \pi_x(C_i^{re})}} \frac{dx \wedge dy \wedge dw}{(x^2 - w)(y^2 - 4xw + g_2x + g_3)} \\
&= c' \int_{\substack{|y^2 - 4xw + g_2x + g_3| = \varepsilon \\ x \in \pi_x(C_i^{re})}} \frac{dx \wedge dy}{(y^2 - 4x^3 + g_2x + g_3)} \\
&= c'' \int_{x \in \pi_x(C_i^{re})} \frac{dx}{y}
\end{aligned}$$

$$
\begin{aligned}
&= c'' \int_{C_i^{re*}} du \qquad (x = \wp(u), y = \wp'(u)) \\
&= \begin{cases} c''(u_{i+1} - u_{i+2}) & (0 \le i \le 6) \\ c''(-u_1 - u_2 - u_3 - u_4) & (i = 7) \end{cases},
\end{aligned}
$$

where C_i^{re*} denotes the pullback of C_i^{re} to the space of u and the last result should be considered modulo $c''(\mathbb{Z} + \mathbb{Z}\tau)$. Since the constant $c \in \mathbb{C}^\times$ has been arbitrary, we can determine it so that $c'' = 1$.

By further blow-up along lines, the simple reflection r_i can be considered to be an exchange of the blow-down structure of $X = X_{P_1,\cdots,P_8}$ and that of $r_i(X) = X_{r_i(P_1,\cdots,P_8)}$, i.e. it just changes how to blow-down corresponding to the change of basis of $\mathrm{Pic}(X)$ (cf. Remark 4.2). Let $u = u_0$ denote the intersection point of a effective divisor D and the curve C. Since the curve C is preserved by r_i (because the modulus of C is not changed), $r_i(D)$ and C also intersect at $u = u_0$. Since D is arbitrary, we have

$$
\chi_X(\alpha) = \chi_{r_i(X)}(r_i(\alpha)) \quad \alpha \in Q(\alpha).
$$

Considering the composition, we have

$$
\chi_X(\alpha) \;=\; \chi_{w(X)}(w(\alpha)) \tag{30}
$$

for all $w \in W(E_7^{(1)})$.

Remark 4.2. In the terminology of [3], this fact means that r_i is a pseudo-isomorphism from X to $r_i(X)$ and determines an exchange of the points of the blow-ups.

From (7),(30), we have

$$
\begin{aligned}
r_i :&\quad (u_{i+1}, u_{i+2}) \mapsto (u_{i+2}, u_{i+1}) \qquad (\text{for } 0 \le i \le 6) \\
r_7 :&\quad (u_1, \cdots, u_8) \mapsto (u_1 - \lambda_1, \cdots, u_4 - \lambda_1, u_5 + \lambda_1, \cdots, u_8 + \lambda_1),
\end{aligned} \tag{31}
$$

where $\lambda_1 = \frac{1}{2}(u_1 + u_2 + u_3 + u_4)$ (preserved elements are omitted). Moreover, since r_i acts on the elliptic curve C birationally and therefore it is a translation for the points on C except P_j and $r_i(P_j)$ $(1 \le j \le 8)$, we have

$$
\begin{aligned}
r_i :&\quad u = 0 \mapsto u = 0 \quad (\text{for } 0 \le i \le 6) \\
r_7 :&\quad u = 0 \mapsto u = \lambda_1 \quad .
\end{aligned} \tag{32}
$$

In case (i-2)

We have

$$
\begin{aligned}
\chi_X(\alpha_i) &= c'' \int_{C_i^{re*}} \frac{du}{u} \\
&= \begin{cases} c'' \log \frac{u_{i+1}}{u_{i+2}} & (0 \le i \le 6) \\ -c'' \log(u_1 u_2 u_3 u_4) & (i = 7) \end{cases}
\end{aligned}
$$

$$
r_i :\quad (u_{i+1}, u_{i+2}) \mapsto (u_{i+2}, u_{i+1}) \qquad (\text{for } 0 \le i \le 6)
$$

$$
r_7 :\quad (u_1, \cdots, u_8) \mapsto (u_1 \lambda_1^{-1}, \cdots, u_4 \lambda_1^{-1}, u_5 \lambda_1, \cdots, u_8 \lambda_1),
$$

where $\lambda_1 = (u_1 u_2 u_3 u_4)^{1/2}$, and we have

$$
\begin{aligned}
r_i :&\quad u = 1 \mapsto u = 1 \quad (\text{for } 0 \le i \le 6) \\
r_7 :&\quad u = 1 \mapsto u = \lambda_1 \quad .
\end{aligned}
$$

In case (i-3)

It is the same with case (i-1).

The action on $\mathbb{P}^3$

We investigate the action to generic points in $X_{P_1,\cdots,P_8}$.

The action of r_i $(0 \leq i \leq 6)$

Since the simple reflection r_i just exchanges the blow-up points, it acts on $\mathbb{P}^3$ as the identical map

$$r_i : \quad \mathbf{x} \mapsto \mathbf{x}. \tag{33}$$

The action of r_7

We write $P_i = (f_x(u_i), f_y(u_i), 1, f_w(u_i))^t$ by the parametric representation of C: (15),(16) or (17). Let $\overline{*}$ denote the image of $*$ by r_7 and set

$$((P_1, P_2, P_3, P_4), (P_5, P_6, P_7, P_8)) = (A, B).$$

For a matrix P, let $1/P$ denote a matrix whose elements are the reciprocal number of corresponding elements of P. The action of r_7 is given by

$$\begin{array}{rl} & ((A,B),\mathbf{x}) \\ \xrightarrow{A^{-1}} & ((\mathrm{Id}, A^{-1}B), A^{-1}\mathbf{x}) \\ \xrightarrow{\mathrm{SCT}} & ((\mathrm{Id}, 1/(A^{-1}B)), 1/(A^{-1}\mathbf{x})) \\ \xrightarrow{\mathrm{diag}\in PGL(4)} & \mathrm{diag}(b_1,b_2,b_3,b_4)((\mathrm{Id}, 1/(A^{-1}B)), 1/(A^{-1}\mathbf{x})) \\ \xrightarrow{\overline{A}} & ((\overline{A},\overline{B}),\overline{\mathbf{x}}), \end{array}$$

where SCT denotes the standard Cremona transformation. Setting $\mathbf{x}' = \mathrm{diag}(b_1,b_2,b_3,b_4)(1/(A^{-1}\mathbf{x}))$, we have

$$\mathbf{x}' = \left(\frac{b_1}{|\mathbf{x}, P_2, P_3, P_4|}, \frac{-b_2}{|\mathbf{x}, P_3, P_4, P_1|}, \frac{b_3}{|\mathbf{x}, P_4, P_1, P_2|}, \frac{-b_4}{|\mathbf{x}, P_1, P_2, P_3|} \right)$$

and

$$\mathbf{x}' = \overline{A}^{-1}\overline{\mathbf{x}} = \left(\left|\overline{\mathbf{x}}, \overline{P_2}, \overline{P_3}, \overline{P_4}\right|, -\left|\overline{\mathbf{x}}, \overline{P_3}, \overline{P_4}, \overline{P_1}\right|, \left|\overline{\mathbf{x}}, \overline{P_4}, \overline{P_1}, \overline{P_2}\right|, -\left|\overline{\mathbf{x}}, \overline{P_1}, \overline{P_2}, \overline{P_3}\right| \right).$$

On the other hand, from (32), in case (i-1) and case (i-2) we have

$$\overline{\mathbf{p}} \;=\; (f_x(\lambda_1) : f_y(\lambda_1) : 1 : f_w(\lambda_1))^t \tag{34}$$

for $\mathbf{p} = (f_x(0) : f_y(0) : 1 : f_w(0))^t = (0:0:0:1)^t$ (in case (i-3) we have $\overline{\mathbf{p}} = (f_x(\lambda_1) : f_y(\lambda_1) : 1 : f_w(\lambda_1))^t$ for $\mathbf{p} = (f_x(1) : f_y(1) : 1 : f_w(1))^t = (0:0:0:1)^t$). Using these, we can obtain b_i explicitly. Consequently, we have

$$\overline{\mathbf{x}} \;=\; \begin{pmatrix} f_x(\breve{u_1}) & f_x(\breve{u_2}) & f_x(\breve{u_3}) & f_x(\breve{u_4}) \\ f_y(\breve{u_1}) & f_y(\breve{u_2}) & f_y(\breve{u_3}) & f_y(\breve{u_4}) \\ 1 & 1 & 1 & 1 \\ f_w(\breve{u_1}) & f_w(\breve{u_2}) & f_w(\breve{u_3}) & f_w(\breve{u_4}) \end{pmatrix} \begin{pmatrix} l_{2,3,4}(\mathbf{x}) \\ -l_{3,4,1}(\mathbf{x}) \\ l_{4,1,2}(\mathbf{x}) \\ -l_{1,2,3}(\mathbf{x}) \end{pmatrix}, \tag{35}$$

where $\check{u_k} := \overline{u_k} = u_k - \lambda_1$ $(1 \le k \le 4)$ ($\check{u_k} := \overline{u_k} = u_k\lambda_1^{-1}$ $(1 \le k \le 4)$ in case (i-2)) and

$$l_{i,j,k}(\mathbf{x}) = \frac{\begin{vmatrix} f_x(\lambda_1) & f_x(\check{u_i}) & f_x(\check{u_j}) & f_x(\check{u_k}) \\ f_y(\lambda_1) & f_y(\check{u_i}) & f_y(\check{u_j}) & f_y(\check{u_k}) \\ 1 & 1 & 1 & 1 \\ f_w(\lambda_1) & f_w(\check{u_i}) & f_w(\check{u_j}) & f_w(\check{u_k}) \end{vmatrix} \begin{vmatrix} 0 & f_x(u_i) & f_x(u_j) & f_x(u_k) \\ 0 & f_y(u_i) & f_y(u_j) & f_y(u_k) \\ 0 & 1 & 1 & 1 \\ 1 & f_w(u_i) & f_w(u_j) & f_w(u_k) \end{vmatrix}}{\begin{vmatrix} x & f_x(u_i) & f_x(u_j) & f_x(u_k) \\ y & f_y(u_i) & f_y(u_j) & f_y(u_k) \\ z & 1 & 1 & 1 \\ w & f_w(u_i) & f_w(u_j) & f_w(u_k) \end{vmatrix}}. \quad (36)$$

5 Dynamical systems

In this section we consider a dynamical system corresponding to a translation of the Weyl group $W(E_7^{(1)})$. Note that although one can consider dynamical systems for all translations, many of them are generated by birational conjugates of one of them.

Notice that

$$r_{w(\alpha_i)}(\beta) := \beta - 2\frac{(w(\alpha_i), \beta)}{(\alpha_i, \alpha_i)} w(\alpha_i) = w^{-1} \circ r_i \circ w(\beta)$$

holds for $w \in W(E_7^{(1)})$, a simple reflection α_i and $\beta \in Q(\alpha)$. Since the map

$$T := r_{E-E_5-E_6-E_7-E_8} \circ r_{E-E_1-E_2-E_3-E_4} \quad (37)$$

acts on the root basis as

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7) \mapsto (\alpha_0, \alpha_1, \alpha_2, \alpha_3 + \delta, \alpha_4, \alpha_5, \alpha_6, \alpha_7 - 2\delta),$$

T is a translation. Therefore T^n defines a birational dynamical system on $X(4,8) \times \mathbb{P}^3$. It can also be considered to be a dynamical system on $\mathbb{P}^3$ with the parameters u_i's (or P_i's).

In case (i-1) or (i-3), similar to the above section, we have the action of T on the parameter space $\{u_i\}$ as

$$T : (u_1, \cdots, u_8) \mapsto (u_1 + \lambda, u_2 + \lambda, u_3 + \lambda, u_4 + \lambda, u_5 - \lambda, \cdots, u_8 - \lambda),$$

where $\lambda = \frac{1}{2}\sum_{i=1}^{8} u_i$.

Since the explicit action of the transformation on $\mathbb{P}^3$ is complicated, we give it using a decomposition. Although it is enough to compose $r_{E-E_5-E_6-E_7-E_8}$ and $r_{E-E_1-E_2-E_3-E_4}$ of course, here we use the fact T can be written as $T = S^2$ by

$$S := r_{E_4-E_8} \circ r_{E_3-E_7} \circ r_{E_2-E_6} \circ r_{E_1-E_5} \circ r_{E-E_1-E_2-E_3-E_4} \quad (38)$$

and describe the action of S. S acts on $\{u_i\}$ as

$$S : (u_1, \cdots, u_8) \mapsto (u_5 + \lambda_1, u_6 + \lambda_1, u_7 + \lambda_1, u_8 + \lambda_1, u_1 - \lambda_1, \cdots, u_4 - \lambda_1) \quad (39)$$

and the action on $\mathbb{P}^3$ is given by (35) and (36), where $\check{u_k} := u_k - \lambda_1$ $(1 \le k \le 4)$.

Along the same line, in case (i-2) we have

$$S : (u_1, \cdots, u_8) \mapsto (u_5\lambda_1, u_6\lambda_1, u_7\lambda_1, u_8\lambda_1, u_1\lambda_1^{-1}, \cdots, u_4\lambda_1^{-1})$$

and the action of S on $\mathbb{P}^3$ is given by (35) and (36), where $\check{u_k} := \overline{u_k} = u_k\lambda_1^{-1}$ $(1 \le k \le 4)$.

6 Conservation law

In this section we prove the following theorem.

Theorem 6.1. *In case* (i) *of Theorem 3.1, the action of the Weyl group preserves each member of the pencil of quadratic surfaces in Theorem 3.1.*

Remark 6.1. The pencil of Theorem 3.1 is given by the linear system $|-\frac{1}{2}K_X|$ in generic. By Theorem 6.1, if the dimension of $|-\frac{1}{2}K_X|$ is 2 or more ($\delta = 0$ by Theorem 3.2), each fiber is preserved by translations associated with the Weyl group $W(E_7^{(1)})$, because (i) each surface is an elliptic surface (ii) each map is birational (hence continuous except at the indefinite points) and preserves the fibration (iii) the modulus of elliptic curve is preserved (iv) at least the intersection curve of the pencil is preserved.

Since for every discrete Painlevé equation the polynomial degree of the n-th iterate is in the order n^2 as $n \to \infty$ [17], by Theorem 6.1 the following corollary follows.

Corollary 6.1. *Let φ be a map on $\mathbb{P}^3$ associated with a translation of $W(E_7^{(1)})$. The degree of φ^n is in the order n^2 as $n \to \infty$.*

Since $-\frac{1}{2}K_X$ is preserved and therefore the pencil itself is preserved, in order to prove Theorem 6.1 it is enough to show that the automorphism of $\mathbb{P}^1$ defined by this correspondence is the identity. In case (i-2) and (i-3) it is easily shown by direct computation. Moreover, the simple reflection $r_i (0 \le i \le 6)$ acts on $\mathbb{P}^3$ as the identity. Hence it is enough to show for r_7 in case (i-1). For r_7, to prove by direct calculation seems to be beyond our computational ability. We prove the theorem in this case by means of a birational representation of $W(E_8^{(1)})$ on $\mathbb{P}^1 \times \mathbb{P}^1$.

Notice that a smooth quadratic surface is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ via the Segré map

$$\mathbb{P}^1 \times \mathbb{P}^1 \ni (x:1, y:1) \mapsto (x:y:1:xy) \in \{(x:y:z:w) \in \mathbb{P}^3 \mid xy - zw = 0\}.$$

Here we reparametrize the parameter space so that the 8 points are on the intersection curve of the pencil spanned by the 2 quadratic surfaces

$$\begin{cases} xy - zw = 0 & (G) \\ (x+y+z)(4w - \frac{g_3}{\wp^3(2t)}z) = \left(w + x + y + \frac{g_2}{4\wp^2(2t)}z\right)^2 & (H) \end{cases}, \tag{40}$$

where $t \in (\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)) \setminus \{0\}$ is an arbitrary extra-parameter.

Remark 6.2. The parameter $\tau = w_2/w_1$ is invariant with respect to t.

By this parametrization, P_i can be parameterized as

$$P_i\left(\frac{\wp(t+u_i)}{\wp(2t)} : \frac{\wp(t-u_i)}{\wp(2t)} : 1 : \frac{\wp(t+u_i)\wp(t-u_i)}{\wp^2(2t)}\right).$$

The action of r_7 on $(\mathbf{u}, t)$ and on $\mathbb{P}^3$ are also given by (31) and (35) respectively, where we set

$$f_x(u) = \frac{\wp(t+u)}{\wp(2t)}, \quad f_y(u) = \frac{\wp(t-u)}{\wp(2t)}, \quad f_w(u) = \frac{\wp(t+u)\wp(t-u)}{\wp^2(2t)},$$

$\breve{u_k} := u_k - \lambda_1$ $(1 \le k \le 4)$ and

$$l_{i,j,k}(\mathbf{x}) = \frac{\begin{vmatrix} f_x(\lambda_1) & f_x(\breve{u_i}) & f_x(\breve{u_j}) & f_x(\breve{u_k}) \\ f_y(\lambda_1) & f_y(\breve{u_i}) & f_y(\breve{u_j}) & f_y(\breve{u_k}) \\ 1 & 1 & 1 & 1 \\ f_w(\lambda_1) & f_w(\breve{u_i}) & f_w(\breve{u_j}) & f_w(\breve{u_k}) \end{vmatrix} \begin{vmatrix} 1 & f_x(u_i) & f_x(u_j) & f_x(u_k) \\ 1 & f_y(u_i) & f_y(u_j) & f_y(u_k) \\ 1 & 1 & 1 & 1 \\ 1 & f_w(u_i) & f_w(u_j) & f_w(u_k) \end{vmatrix}}{\begin{vmatrix} x & f_x(u_i) & f_x(u_j) & f_x(u_k) \\ y & f_y(u_i) & f_y(u_j) & f_y(u_k) \\ z & 1 & 1 & 1 \\ w & f_w(u_i) & f_w(u_j) & f_w(u_k) \end{vmatrix}}.$$

Next, we list the necessary results by Murata et al. [11] concerning birational maps on $\mathbb{P}^1 \times \mathbb{P}^1$ whose space of initial conditions $S = S_{P_1,\cdots,P_8}$'s are given by blow-ups of $\mathbb{P}^1 \times \mathbb{P}^1$ at generic 8 points $P_1, P_2, \cdots, P_8$ on the smooth curve of degree $(2,2)$

$$(x+y+1)(4xy - \frac{g_3}{\wp^3(2t)}) = \left(xy + x + y + \frac{g_2}{4\wp^2(2t)}\right)^2. \tag{41}$$

Let x and y be the usual coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$. Let H_0, H_1 and E_i denote the total transform of $x = c \in \mathbb{P}^1$, that of $y = c' \in \mathbb{P}^1$ and that of the exceptional divisor generated by the blow-up at the point P_i respectively. The Picard group and the canonical divisor of the surface S are

$$\begin{aligned} \mathrm{Pic}(S) &= \mathbb{Z}H_0 \oplus \mathbb{Z}H_1 \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \cdots \oplus \mathbb{Z}E_8 \\ K_S &= -2H_0 - 2H_1 + E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7 + E_8 \end{aligned}$$

and the intersection numbers are given by

$$H_i \cdot H_j = 1 - \delta_{i,j}, \quad H_i \cdot E_j = 0, \quad (E_i, E_j) = -\delta_{i,j}.$$

The root basis is given by

$$\begin{aligned} &\alpha_i = E_{7-i} - E_{8-i} \quad (i = 0, 1, \cdots, 5), \\ &\alpha_6 = H_1 - E_1 - E_2, \quad \alpha_7 = H_0 - H_1, \quad \alpha_8 = E_1 - E_2, \end{aligned}$$

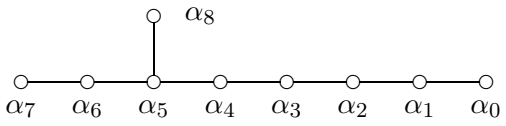

Figure 2: $E_8^{(1)}$ Dynkin diagram

and each action on the parameter space $(\mathbf{u}, t)$ becomes

$$\begin{aligned} &r_i : (u_{7-i}, u_{8-i}) \mapsto (u_{8-i}, u_{7-i}), \qquad (i = 0, 1, \cdots, 5) \\ &r_6 : \left(\begin{matrix} u_1 & u_2 & u_3 & u_4 \\ u_5 & u_6 & u_7 & u_8 \end{matrix}, t \right) \\ &\mapsto \left(\begin{matrix} u_1 - 3\lambda_{1,2} & u_2 - 3\lambda_{1,2} & u_3 + \lambda_{1,2} & u_4 + \lambda_{1,2} \\ u_5 + \lambda_{1,2} & u_6 + \lambda_{1,2} & u_7 + \lambda_{1,2} & u_8 + \lambda_{1,2} \end{matrix}, t - \lambda_{1,2} \right), \\ &r_7 : t \mapsto -t, \qquad r_8 : (u_1, u_2) \mapsto (u_2, u_1), \end{aligned} \tag{42}$$

where $\lambda_{1,2} = \frac{1}{4}(2t + u_1 + u_2)$.

Proof of Theorem 6.1
As mentioned above, it is sufficient to show for $r_7 = r_{E-E_1-E_2-E_3-E_4}$ in case (i-1). Let G and H denote the matrices corresponding to (G) and (H) of the pencil (40). We write $(\overline{x} : \overline{y} : \overline{z} : \overline{w}) := r_7(x : y : z : w)$. Since the image of a member of the pencil $(\overline{x} : \overline{y} : \overline{z} : \overline{w})|_{\mathbf{x}^t(\alpha_0 G+\alpha_1 H)\mathbf{x}=0}$ $((\alpha_0 : \alpha_1) \in \mathbb{P}^1)$ is again an element of $|-\frac{1}{2}K_X|$ and therefore is again a member of the pencil (the intersection curve does not move). Hence, there exists $(\beta_0 : \beta_1) \in \mathbb{P}^1$ such that $\overline{\mathbf{x}}^t(\beta_0 G+\beta_1 H)\overline{\mathbf{x}} = 0$. Since this correspondence defines an automorphism of the base space $\mathbb{P}^1$ of the pencil, it is enough to show that 3 members of the pencil are preserved. Hence we may assume $\mathrm{rank}(\alpha_0 G+\alpha_1 H) = \mathrm{rank}(\beta_0 G+\beta_1 H) = 4$. In the following, we assume $(\alpha_0 : \alpha_1) \neq (\beta_0 : \beta_1)$ and lead a contradiction.

Lemma 6.1. *Assume* $\mathrm{rank}(\alpha_0 G + \alpha_1 H) = 4$. *There exist* $P \in PGL(4), v \in \mathbb{C}^\times$ *and* $t' \in (\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)) \setminus \{0\}$ *such that*

$$P^t(\alpha_0 G + \alpha_1 H)P = G, \quad P^t H P = vH(\wp(2t'))$$

and moreover there exist $Q_1, Q_2 \in PGL(4), v_1, v_2 \in \mathbb{C}^\times$ *and* $c, d \in \mathbb{C} \setminus \{0, 1\}$ *such that*

$$\begin{aligned} Q_1^t(\alpha_0 G + \alpha_1 H)Q_1 = \mathrm{diag}(1,1,1,1), &\quad Q_1^t H Q_1 = v_1 \mathrm{diag}(0,1,c,d), \\ Q_2^t G Q_2 = \mathrm{diag}(1,1,1,1), &\quad Q_2^t H(\wp(2t'))Q_2 = v_2 \mathrm{diag}(0,1,c,d). \end{aligned}$$

Proof. Since $\mathrm{rank}H = 3$, similar to the proof of Theorem 3.1, G and H can be transformed to

$$G' = \mathrm{Id.}, \quad H' = v\mathrm{diag}(0, a, b, 1) \quad a, b, 1 \text{ are different each other}$$

by $PGL(4)$. Here, a and b are functions of g_2, g_3 and $\wp(2t)$. Conversely, if $a, b, 1$ are different each other, there exist corresponding g_2, g_3 and $\wp(2t)$. On the other hand, they are also transformed to

$$(\alpha_0 G + \alpha_1 H)'' = \mathrm{Id.} \quad H'' = v_1 \mathrm{diag}(0, c, d, 1) \quad c, d, 1 \text{ are different each other}$$

by $PGL(4)$. Hence, there exist corresponding $g_2', g_3', \wp(2t')$ and $P \in PGL(4)$ such that

$$P : \alpha_0 G + \alpha_1 H, H \mapsto G, H(g_2', g_3', \wp(2t')).$$

Here, r_7 defines a birational map between the intersection curves and therefore the parameter $\tau = w_2/w_1$ does not change. As Remark 6.2, if we fix w_1 as $w_1 = 1$, only t' can change, which shows that H' is a function depending only on $\wp(2t')$. □

For $(\alpha_0 : \alpha_1)$ and $(\beta_0 : \beta_1)$ we denote t' determined by Lemma 6.1 by t_a and t_b respectively. First, we show $t_a \neq t_b$. Assume $t_a = t_b$. Since only $PGL(4)$ is needed for $\mathrm{diag}(0,1,c,d)$ when one normalizes the pair $\mathrm{diag}(1,1,1,1), \mathrm{diag}(0,1,c,d)$ to the pair $\mathrm{diag}(0,1,1,1), \mathrm{diag}(1,1,\lambda,0)$, there exist members J and J' of the pencil, where $J \neq J', \mathrm{rank}J = \mathrm{rank}J' = 3$, such that both pairs J, H and J', H can be transformed to the form $\mathrm{diag}(0,1,1,1), \mathrm{diag}(1,1,\lambda,0)$ by $PGL(4)$. Hence there exist members $K = \mathrm{diag}(0,1,1,1)$ and K' of the pencil $\{s_0\mathrm{diag}(0,1,1,1) + s_1\mathrm{diag}(1,1,\lambda,0)\}$, where $K \neq K', \mathrm{rank}K' = 3$, such that the pair K, H can be transformed to the pair K', H by $PGL(4)$. The members of this pencil in rank 3 are $K = \mathrm{diag}(0,1,1,1), \mathrm{diag}(1,1,\lambda,0), K_1 := \mathrm{diag}(1,0,\lambda-1,-1)$ and $K_2 := \mathrm{diag}(-1,\lambda-1,0,\lambda)$ only. Therefore, K' should be K_1 or K_2. Normalizing each by $PGL(4)$, we have that λ is $\lambda/(\lambda-1), (\lambda-1)/\lambda, 1-\lambda$ or $1/(1-\lambda)$. If

$\lambda \neq 2, \frac{1}{2}, \frac{1\pm 3I}{2}$, it does not coincides with λ. Hence, if τ does not correspond to these λ, we have $t_a \neq t_b$ (as shown by (27), λ does not depend on $\wp(2t)$)

Let P_a, P_b denote the elements of $PGL(4)$ which give t_a, t_b in Lemma 6.1. We consider the map defined by $P_b \circ r_7 \circ P_a^{-1}$. Since the period map χ_X is conserved by $PGL(4)$, we have $\overline{u_i} = u_i - \lambda_1 + c$ $(1 \leq i \leq 4)$, $\overline{u_i} = u_i + \lambda_1 + c$ $(5 \leq i \leq 8)$, where c is a constant. On the other hand, the map

$$\mathbf{x}^t G\mathbf{x} = 0 \xrightarrow{P_a^{-1}} \mathbf{x}^t(\alpha_0 G + \alpha_1 H)\mathbf{x} = 0 \xrightarrow{r_7} \mathbf{x}^t(\beta_0 G + \beta_1 H)\mathbf{x} = 0 \xrightarrow{P_b} \mathbf{x}^t G\mathbf{x} = 0$$

and Segré map define a birational map on $\mathbb{P}^1 \times \mathbb{P}^1$.

By blow-up $\mathbb{P}^1 \times \mathbb{P}^1$ at $\{P_i\}$ and $\{\overline{P_i}\}$ the map is lifted to an isomorphism and included by the Weyl group of type $E_8^{(1)}$. Since the Weyl group conserves $\chi_S(\delta) = -\sum_{i=1}^8 u_i$, we have $c = 0$. Hence r_7 can be written as

$$r_7 = r_{H_0+H_1-E_1-E_2-E_3-E_4} = r_6 \circ r_7 \circ r_5 \circ r_8 \circ r_4 \circ r_5 \circ r_6 \circ r_5 \circ r_4 \circ r_8 \circ r_5 \circ r_7 \circ r_6$$

by the root system of $E_8^{(1)}$, and we have $t_a = t_b$ from (42), which contradicts $t_a \neq t_b$.

We have shown $(\alpha_0 : \alpha_1) = (\beta_0 : \beta_1)$ if τ does not correspond to $\lambda \neq 2, \frac{1}{2}$ or $\frac{1\pm 3I}{2}$. When $\lambda = 2, \frac{1}{2}, \frac{1\pm 3I}{2}$ it can be shown by continuity of $\wp$ with respect to τ.

□

7 Conclusions and discussions

In this paper, we defined the inner product for the Picard group of varieties obtained by blow-ups at 8 points in $\mathbb{P}^3$ by means of the intersection numbers and the anti-canonical divisor and showed that the symmetry group defined by means of the inner product is the Weyl group of type $E_7^{(1)}$. As in 2-dimensional case [15], if the configuration of points is special, the symmetry may become smaller.

This method can be applied to other families of 3-dimensional rational varieties.

Example 7.1. Let X be a variety obtained by blow-ups at generic 6 points in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ and let H_i and E_i denote the total transform of divisor class of a plane such that one of its coordinate of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is constant and that of the exceptional divisor generated by a blow-up respectively. The symmetry group $\mathrm{Gr}(\{X\})$ becomes the Weyl group of type $E_7^{(1)}$ defined by the root system

$$\alpha_0 = H_0 - H_2,\ \alpha_1 = H_1 - H_2,\ \alpha_3 = H_2 - E_1 - E_2,$$
$$\alpha_i = E_{i-1} - E_i\ (3 \leq i \leq 6),\quad \alpha_7 = E_1 - E_2,$$

where the inner product is given by $(H_i, H_j) = 1 - \delta_{i,j}$, $(H_i, E_j) = 0, (E_i, E_j) = -\delta_{i,j}$. This X and the variety obtained of blow-ups at generic 8 points in $\mathbb{P}^3$ are not isomorphic. Thus, the relation between these 2 Weyl groups is not trivial.

It may be worth commenting that the space of initial conditions for Kajiwara-Noumi-Yamada's birational representation of the Weyl group of type $A_1^{(1)} \times A_2^{(1)}$ [9] can be obtained when the points of blow-ups are in special position in the above example. These examples are 3-dimensional but it is expected that our method can be applied to 4 (or higher)-dimensional cases.

The other results are summarized as follows.

· We parametrized the configuration space by normalizing the pencil of quadratic surfaces passing through 8 points ($\in |-\frac{1}{2}K_X|$) by $PGL(4)$. The intersection curve is an elliptic curve in generic.
· In order to obtain concrete expression of the action of the Weyl group, we introduced a 3-dimensional analogue of period map.
· We showed the action of the Weyl group preserves each element of $|-\frac{1}{2}K_X|$ and therefore it reduces to the action on $\mathbb{P}^1 \times \mathbb{P}^1$. The reduced action of the Weyl group of type $E_7^{(1)}$ is included by the action of the Weyl group of type $E_8^{(1)}$, which is the symmetry of the family of generic Halphen surfaces.

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