微分積分 II　期末試験解答　(2009 年度, 担当: 関口 良行)

<u>計算過程も記述すること</u>

1. 積分せよ.

(1) $\displaystyle\int \frac{x^3-3}{x^3-x^2-x+1}\,dx$

$$= \int \left\{1 + \frac{-1}{x+1} + \frac{2}{x-1} + \frac{-1}{(x-1)^2}\right\} dx$$
$$= x - \log|x+1| + 2\log|x-1| + \frac{1}{x-1} + C$$
$$= x + \frac{1}{x-1} + \log\frac{(x-1)^2}{|x+1|} + C$$

(2) $\displaystyle\int \frac{1}{2+\sin x}\,dx$

$\tan\dfrac{x}{2} = t$ とおくと

$$\int \frac{1}{2+\sin x}\,dx = \int \frac{1}{2+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt$$
$$= \int \frac{1}{t^2+t+1}\,dt = \int \frac{1}{\left(t+\frac{1}{2}\right)^2+\frac{3}{4}}\,dt$$
$$= \int \frac{1}{\frac{3}{4}\left\{1+\left(\frac{2}{\sqrt{3}}t+\frac{1}{\sqrt{3}}\right)^2\right\}}\,dx$$
$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}\tan\frac{x}{2}+\frac{1}{\sqrt{3}}\right) + C$$

(3) $\displaystyle\int_1^\infty \frac{dx}{x(1+x^2)}$

$$= \int \left(\frac{1}{x} - \frac{x}{1+x^2}\right) dx$$
$$= \left[\log|x| - \frac{1}{2}\log|1+x^2|\right]_1^\infty$$
$$= \left[\frac{1}{2}\log\frac{x^2}{1+x^2}\right]_1^\infty = \frac{1}{2}\log 2$$

2. 正の定数 $a, b > 0$ に対して, 曲線 $y = \dfrac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right)$ (懸垂線と呼ばれる) の $-b \le x \le b$ の部分の長さを求めよ.

$$(\text{長さ}) = \int_{-b}^{b} \sqrt{1 + \frac{1}{4}\left(e^{x/a} - e^{-x/a}\right)^2}\,dx$$
$$= \int_{-b}^{b} \frac{1}{2}\left(e^{x/a} + e^{-x/a}\right) dx$$
$$= a\left(e^{b/a} - e^{-b/a}\right)$$

3. 偏微分に関する次の問いに答えよ.

(1) $f(x,y)=\sin^{-1}\dfrac{y}{x}$ $(x>0)$ に対して, f_x, f_y を求めよ.

$$f_x=\frac{-y}{x\sqrt{x^2-y^2}}$$
$$f_y=\frac{1}{\sqrt{x^2-y^2}}$$

(2) $f(x,y)=x\cos\sqrt{y}$, $x(u,v)=uv$, $y(u,v)=u^2+v^2$ に対して,
合成関数 $z(u,v)=f(x(u,v),y(u,v))$ の偏導関数 z_u, z_v を求めよ.

$$f_x=\cos\sqrt{y}, f_y=-\frac{x}{2\sqrt{y}}\sin\sqrt{y}$$
$$x_u=v, x_v=u, \quad y_u=2u, y_v=2v$$
$$f_u=f_xx_u+f_yy_u=v\left(\cos\sqrt{u^2+v^2}-\frac{u^2\sin\sqrt{u^2+v^2}}{\sqrt{u^2+v^2}}\right)$$
$$f_v=f_xx_v+f_yy_v=u\left(\cos\sqrt{u^2+v^2}-\frac{v^2\sin\sqrt{u^2+v^2}}{\sqrt{u^2+v^2}}\right)$$

4. 次の函数の極値を求めよ.

$$f(x,y)=x^3+2x^2+xy+y^2+x+2y+3$$

$f_x=3x^2+4x+y+1=0, \quad f_y=x+2y+2=0$ を満たす (x,y) を求める.
$y=\dfrac{-1}{2}x-1$ を代入し, $3x^2+4x-\dfrac{1}{2}x-1+1=0, x(6x+7)=0$ より
$(x,y)=(0,-1), \quad \left(-\dfrac{7}{6},-\dfrac{5}{12}\right)$.
$\det\left(\nabla^2f(x,y)\right)=\det\begin{pmatrix}6x+4 & 1\\ 1 & 2\end{pmatrix}=12x+7$
$(0,1)$ において ,$\det\left(\nabla^2f(0,-1)\right)>0, f_{xx}(0,-1)>0,$ なので極小. $f(0,-1)=2.$
$\left(-\dfrac{7}{6},-\dfrac{5}{12}\right)$ において ,$\det\left(\nabla^2f(-7/6,-5/12)\right)<0$ なので不定 .

5. 以下の重積分を計算せよ.

(1) $\displaystyle\iint_D x\,dxdy$, ここで D は直線 $y=x+2$ と放物線 $y=x^2$ で囲まれた領域とする.

$$
\begin{aligned}
&D : x^2 \leq y \leq x+2,\ -1 \leq x \leq 2 \\
&\int_{-1}^{2} \left\{ \int_{x^2}^{x+2} x\,dy \right\} dx \\
&= \int_{-1}^{2} x(x+2-x^2)\,dx = \left[\frac{1}{3}x^3 + x^2 - \frac{1}{4}x^4 \right]_{-1}^{2} = \frac{9}{4}
\end{aligned}
$$

(2) $\displaystyle\iint_D \frac{dxdy}{\sqrt{4-x^2-y^2}}$, ここで $D = \{(x,y) \mid 0 \leq x^2+y^2 \leq 3, x \geq 0, x-y \geq 0\}$.

$$x = r\cos\theta,\ y = r\sin\theta \text{ とおくと } \frac{\partial(x,y)}{\partial(r,\theta)} = r$$

$$\int_{-\pi/2}^{\pi/4} \left\{ \int_0^{\sqrt{3}} \frac{r}{\sqrt{4-r^2}}\,dr \right\} d\theta = \int_{-\pi/2}^{\pi/4} \left[-\sqrt{4-r^2} \right]_{r=0}^{\sqrt{3}} dr d\theta = \frac{3}{4}\pi$$

(3) $\displaystyle\iint_D \cos(8x+8y)\,dxdy$, ここで $D = \left\{ (x,y) \mid 0 \leq 2x+3y \leq \frac{\pi}{4},\ \pi \leq 2x-y \leq 2\pi \right\}$.

$$
\begin{aligned}
&\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \text{ とおく.} \\
&\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-8} \begin{pmatrix} -1 & -3 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad \frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{8} \\
&\int_0^{\pi/4} \left\{ \int_\pi^{2\pi} \cos(u+3v+2u-2v) \left| -\frac{1}{8} \right| dv \right\} du \\
&= \int_0^{\pi/4} \frac{1}{8} \left[\sin(3u+v) \right]_{v=\pi}^{2\pi} du = \int_0^{\pi/4} \frac{1}{8} \left\{ \sin(3u+2\pi) - \sin(3u+\pi) \right\} du \\
&= \frac{1}{24} \left[-\cos(3u+2\pi) + \cos(3u+\pi) \right]_0^{\pi/4} \\
&= \frac{1}{24} \left\{ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 1 + 1 \right\} = \frac{\sqrt{2}+2}{24} = \frac{1}{12}\left(1 + \frac{1}{\sqrt{2}} \right).
\end{aligned}
$$

(u から先に積分すると)

$$
\begin{aligned}
&\int_\pi^{2\pi} \frac{1}{8} \left[\frac{1}{3}\sin(3u+v) \right]_{u=0}^{\pi/4} dv = \frac{1}{24} \int_\pi^{2\pi} \left\{ \sin\left(\frac{3\pi}{4} + v \right) - \sin v \right\} dv \\
&= \frac{1}{24} \left[-\cos\left(\frac{3\pi}{4} + v \right) + \cos v \right]_\pi^{2\pi} = \frac{\sqrt{2}+2}{24}.
\end{aligned}
$$