線形代数 I: 余因子行列

逆行列の公式

3×3 行列

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

に対して,

$$\tilde{A} = \begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \\ \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix} = \begin{bmatrix} \varDelta_{11} & \varDelta_{21} & \varDelta_{31} \\ \varDelta_{12} & \varDelta_{22} & \varDelta_{32} \\ \varDelta_{13} & \varDelta_{23} & \varDelta_{33} \end{bmatrix}$$

とおく. この $\tilde{A}$ を A の余因子行列と呼び, $\varDelta_{11}$ を A の $(1,1)$ 余因子, $\varDelta_{12}$ を A の $(1,2)$ 余因子などと呼ぶ. ここで, (i,j) 余因子の符号は $(-1)^{i+j}$ に等しい. 例えば, $(2,1)$ 余因子 $\varDelta_{21}$ の符号は $(-1)^{2+1} = -1$ である.

定理 (逆行列の公式)**.** $|A| \neq 0$ ならば,

$$A^{-1} = \frac{1}{|A|}\tilde{A}$$

が成り立つ.

例.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 2 & 1 & -5 \end{bmatrix}$$

の逆行列を公式を用いて求める. 余因子はそれぞれ

$$\varDelta_{11} = (-1)^{(1+1)} \begin{vmatrix} 1 & 0 \\ 1 & -5 \end{vmatrix}, \ \varDelta_{21} = (-1)^{(2+1)} \begin{vmatrix} 0 & -3 \\ 1 & -5 \end{vmatrix}, \varDelta_{31} = (-1)^{(3+1)} \begin{vmatrix} 0 & -3 \\ 1 & 0 \end{vmatrix},$$
$$\varDelta_{12} = (-1)^{(1+2)} \begin{vmatrix} 0 & 0 \\ 2 & -5 \end{vmatrix}, \varDelta_{22} = (-1)^{(2+2)} \begin{vmatrix} 1 & -3 \\ 2 & -5 \end{vmatrix}, \varDelta_{32} = (-1)^{(3+2)} \begin{vmatrix} 1 & -3 \\ 0 & 0 \end{vmatrix},$$
$$\varDelta_{13} = (-1)^{(1+3)} \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix}, \varDelta_{23} = (-1)^{(2+3)} \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix}, \varDelta_{33} = (-1)^{(3+3)} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

であり, $|A|=1$ なので,

$$A^{-1}=\begin{bmatrix}-5 & -3 & 3\\ 0 & 1 & 0\\ -2 & -1 & 1\end{bmatrix}$$

となる.

(公式の解説).

$$\frac{1}{|A|}\tilde{A}A=\frac{1}{|A|}\begin{bmatrix}\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix} & -\begin{vmatrix}a_{12} & a_{13}\\ a_{32} & a_{33}\end{vmatrix} & \begin{vmatrix}a_{12} & a_{13}\\ a_{22} & a_{23}\end{vmatrix}\\ -\begin{vmatrix}a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix} & \begin{vmatrix}a_{11} & a_{13}\\ a_{31} & a_{33}\end{vmatrix} & -\begin{vmatrix}a_{11} & a_{13}\\ a_{21} & a_{23}\end{vmatrix}\\ \begin{vmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix} & -\begin{vmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{vmatrix} & \begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}\end{bmatrix}\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}$$

$$=\frac{1}{|A|}\begin{bmatrix}a_{11}\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}-a_{21}\begin{vmatrix}a_{12} & a_{13}\\ a_{32} & a_{33}\end{vmatrix}+a_{31}\begin{vmatrix}a_{12} & a_{13}\\ a_{22} & a_{23}\end{vmatrix} & * & *\\ -a_{11}\begin{vmatrix}a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix}+a_{21}\begin{vmatrix}a_{11} & a_{13}\\ a_{31} & a_{33}\end{vmatrix}-a_{31}\begin{vmatrix}a_{11} & a_{13}\\ a_{21} & a_{23}\end{vmatrix} & * & *\\ a_{11}\begin{vmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}-a_{21}\begin{vmatrix}a_{11} & a_{12}\\ a_{31} & a_{32}\end{vmatrix}+a_{31}\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix} & * & *\end{bmatrix}\quad(2,3\text{ 列は省略})$$

となるので, 右辺の 1 行 1 列の要素を計算すると

$$\begin{aligned}&a_{11}\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}-a_{21}\begin{vmatrix}a_{12} & a_{13}\\ a_{32} & a_{33}\end{vmatrix}+a_{31}\begin{vmatrix}a_{12} & a_{13}\\ a_{22} & a_{23}\end{vmatrix}\\ &\quad=\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ 0 & a_{22} & a_{23}\\ 0 & a_{32} & a_{33}\end{vmatrix}+\begin{vmatrix}0 & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ 0 & a_{32} & a_{33}\end{vmatrix}+\begin{vmatrix}0 & a_{12} & a_{13}\\ 0 & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}=\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix}=|A|\end{aligned}$$

となる (余因子展開を用いる). 右辺の 2 行 1 列は

$$\begin{aligned}&-a_{11}\begin{vmatrix}a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix}+a_{21}\begin{vmatrix}a_{11} & a_{13}\\ a_{31} & a_{33}\end{vmatrix}-a_{31}\begin{vmatrix}a_{11} & a_{13}\\ a_{21} & a_{23}\end{vmatrix}\\ &\quad=\begin{vmatrix}a_{11} & a_{11} & a_{13}\\ a_{21} & 0 & a_{23}\\ a_{31} & 0 & a_{33}\end{vmatrix}+\begin{vmatrix}a_{11} & 0 & a_{13}\\ a_{21} & a_{21} & a_{23}\\ a_{31} & 0 & a_{33}\end{vmatrix}+\begin{vmatrix}a_{11} & 0 & a_{13}\\ a_{21} & 0 & a_{23}\\ a_{31} & a_{31} & a_{33}\end{vmatrix}=\begin{vmatrix}a_{11} & a_{11} & a_{13}\\ a_{21} & a_{21} & a_{23}\\ a_{31} & a_{31} & a_{33}\end{vmatrix}=0\end{aligned}$$

となる (同じ行があると行列式は 0) ので, 同様に計算すると,

$$\frac{1}{|A|}\tilde{A}A = \frac{1}{|A|}\begin{bmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{bmatrix} = E$$

が成り立つ. また同様に

$$A\left(\frac{1}{|A|}\tilde{A}\right) = E$$

も示せるので, $\frac{1}{|A|}\tilde{A} = A^{-1}$ である. $\square$